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ALTERNATE
CIRCLES

ALTERNATE CIRCLES

AND

THEIR CONNEXION WITH

THE ELLIPSE.

BY THE

DUKE OF SOMERSET.

LONDON:
HENRY G. BOHN, YORK STREET, COVENT GARDEN.
1851.

183. c. 21.

ADVERTISEMENT.

THE present Volume is a sequel to the Treatise on the ELLIPSE.

It consists of Two Sections.

The First Section contains some Geometrical Propositions, demonstrated by an Algebraical Process.

The Second Section employs Alternate Circles to simplify, or to reduce to lower dimensions, such equations as express the properties of the Ellipse.

ALTERNATE CIRCLES.

Section the First.

PROPOSITION I.

Problem.—*To describe two alternate circles, of which the areas shall be in a given ratio, and their sum equal to a given rectangle.*

Let the given ratio be as $a\,a$ to $b\,b$, and the given rectangle $k\,l$.

Let c be the circumference of a circle of which the diameter is unity.

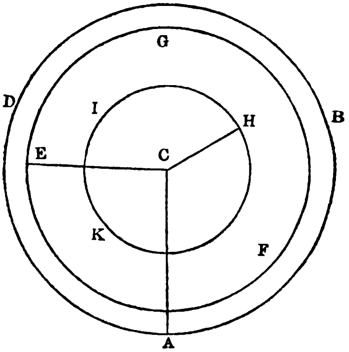
Now, if x be the semi-diameter of a circle, its diameter will be $2\,x$.

Then, $1 : 2x :: c : 2cx$.

Therefore, $2cx$ is the circumference of a circle of which the radius is x.

Then half that circumference is cx.

Now, because the area of a circle is equal to the rectangle contained by the semi-diameter and a straight line equal to half the circumference, cxx will represent a space equal to the area of a circle of which the semi-diameter is x.



Then, let $kl = cxx$ be equal to the area of the external circle.

Let the semi-diameters of the alternate circles be y and z.

Then, because circles are to one another as the squares of their semi-diameters,

$$yy + zz = xx = \frac{kl}{c},$$

And $yy : zz :: aa : bb$.

$$yy = \frac{aazz}{bb}$$

$$yy = \frac{kl}{c} - zz$$

$$\frac{aazz}{bb} = \frac{kl}{c} - zz$$

$$zz = \frac{bbkl}{(aa + bb)c}$$

$$z = \frac{b\sqrt{kl}}{\sqrt{(aa + bb)c}}$$

$$y = \frac{a\sqrt{kl}}{\sqrt{(aa + bb)c}}$$

$$x = \frac{\sqrt{kl}}{\sqrt{c}}.$$

Then, from any point C, at the distance CA, equal to $\frac{\sqrt{kl}}{\sqrt{c}}$, describe the circle ABD.

From the same point, at the distance CE, equal to $\frac{a\sqrt{kl}}{\sqrt{(aa+bb)c}}$, describe the circle EFG.

From the same point, at the distance CH, equal to $\frac{b\sqrt{kl}}{\sqrt{(aa+bb)c}}$, describe the circle HIK.

The two last circles shall be those required.

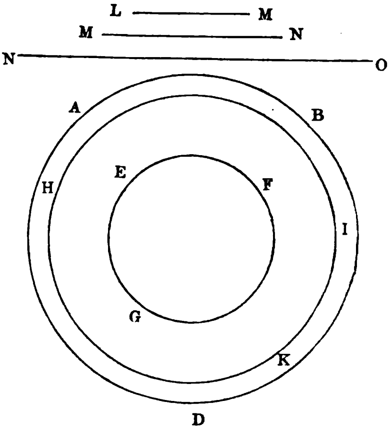
For, because $\frac{aakl}{(aa+bb)c} : \frac{bbkl}{(aa+bb)c} :: aa : bb$, the areas of the alternate circles are, one to the other, in the given ratio.

And because $\frac{aakl}{aa+bb} + \frac{bbkl}{aa+bb} = \frac{(aa+bb)kl}{aa+bb} = kl$, the sum of the areas is equal to the given rectangle.

PROPOSITION II.

THEOREM.—*If three straight lines be harmonical proportionals, the area of one alternate circle be equal to a rectangle contained by the first and second, and the area of the other to a rectangle contained by the second and third; the area of the external circle shall be equal to twice the rectangle contained by the first and the third.*

Let the lines LM, MN, and NO, be harmonical proportionals, let the area of the alternate circle EFG be equal to a rectangle contained by LM and MN, and let the area of the alternate circle HIK be equal to a rectangle contained by MN and NO; the area of the external circle ABD shall be equal to the double of the rectangle contained by LM and NO.



Call LM, a; MN, b; and NO, g.

Then, because a, b, and g are harmonical proportionals,

$$a-b:b-g::a:g,$$
$$ag-bg=ab-ag,$$
$$2ag=ab+bg.$$

SCHOL.—Proposition 1st shows, that, if c be the circumference of a circle, of which the radius is unity, the radius of a circle equal to ab will be $\frac{\sqrt{ab}}{\sqrt{c}}$.

And the radius of a circle equal to bg will be $\frac{\sqrt{bg}}{\sqrt{c}}$.

Now, if these circles be described, the 2d Proposition of the first book of the Treatise on the Ellipse shows how to describe the external circle.

The radius of the external circle will, however, be equal to $\frac{\sqrt{2ag}}{\sqrt{c}}$.

PROPOSITION III.

PROBLEM.—*To describe three concentric circles in arithmetical progression, the two lesser being alternate, and the common difference being given.*

Let dd be the common difference of the terms.

Call the semi-diameters of the circles x, y, and z, and let x be the least, and z the greatest of the three.

Let c be the circumference of a circle, of which the diameter is unity.

Then, as has been shown in the 1st Proposition, cxx will represent a space equal to the area of a circle of which the semi-diameter is x.

And $cxx + dd$ will represent a space equal to the area of a circle of which the semi-diameter is y.

And, $cxx + 2dd$ will represent a space equal to the area of a circle of which the semi-diameter is z.

Now, because, of the three circles, the two lesser are alternate,

$$cxx + 2dd = cxx + dd + cxx$$

$$dd = cxx$$

$$\frac{dd}{c} = xx$$

$$x = \frac{d}{\sqrt{c}}.$$

Now, $cyy = cxx + dd = 2dd$

$$yy = \frac{2dd}{c}$$

$$y = \frac{d\sqrt{2}}{\sqrt{c}}.$$

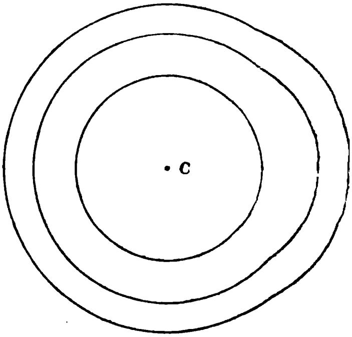
And, $czz = cxx + 2dd = 3dd$

$$zz = \frac{3dd}{c}$$

$$z = \frac{d\sqrt{3}}{\sqrt{c}}.$$

Then, if there be described, from the centre, C, three circles, with semi-diameters respectively equal to $\frac{d}{\sqrt{c}}$, $\frac{d\sqrt{2}}{\sqrt{c}}$, and $\frac{d\sqrt{3}}{\sqrt{c}}$, they shall be those required.

For, the areas of the circles will be equal to dd, $2dd$, and $3dd$, and the double of the middle term is equal to the sum of the two others; wherefore they are in arithmetical progression.



Again, because

$$3dd = 2dd + dd,$$
$$cxx + 2dd = cxx + dd + cxx,$$
$$2dd = cxx + dd,$$
$$cxx + dd = cxx + dd.$$

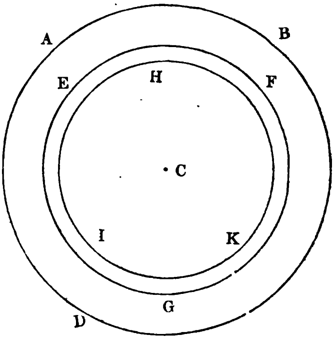
So, the area of the greatest circle is equal to the sum of the areas of the other two.

Therefore the other two are alternate.

PROPOSITION IV.

Problem.—*To find two alternate circles, such that the rectangle contained by their semi-diameters shall be equal to a given square, and that the external circle shall have a given ratio to the difference between their areas.*

Let the rectangle contained by the semi-diameters be equal to aa.



Let the given ratio be that of bb to gg.

Call one of the semi-diameters x and the other y.

Then, because the area of the external circle is equal to the sum of the areas of the alternate circles; and that the areas of circles are, to one another, as the squares of their semi-diameters, $xx + yy : xx - yy :: bb : gg$, and, therefore,

$$(bb + gg)yy = (bb - gg)xx.$$

Again, $xy = aa$, $y = \frac{aa}{x}$, and $yy = \frac{a^4}{x^2}$.

Wherefore, $(bb + gg)\frac{a^4}{x^2} = (bb - gg)x^2$

$$(bb + gg)\,a^4 = (bb - gg)x^4$$

$$x^4 = \frac{a^4(bb + gg)}{bb - gg}$$

$$x^2 = \frac{a^2\sqrt{bb + gg}}{\sqrt{bb - gg}}$$

$$x = \frac{a\sqrt{\sqrt{bb + gg}}}{\sqrt{\sqrt{bb - gg}}}$$

And, $y = \frac{aa}{x} = \frac{a\sqrt[4]{bb - gg}}{\sqrt[4]{bb + gg}}$.

Then, with a radius equal to $\frac{a\sqrt[4]{bb - gg}}{\sqrt[4]{bb + gg}}$, describe a circle, having a centre, *C*, and a circumference, *HIK*.

From the same centre, at the distance $\frac{a\sqrt[4]{bb + gg}}{\sqrt[4]{bb - gg}}$, describe another circle, *EFG*.

From the same centre, with a radius equal to $\frac{ab\sqrt{2}}{\sqrt[4]{b^4 - g^4}}$, describe a third circle, *ABD*.

These three circles shall be those required.

For, the rectangle contained by the semi-diameters of the alternate circles is,

$$\frac{a\sqrt[4]{bb + gg}}{\sqrt[4]{bb - gg}} \times \frac{a\sqrt[4]{bb - gg}}{\sqrt[4]{bb + gg}} = aa.$$

And the sum of the squares of the semi-diameters is

$$\frac{a^2\sqrt{bb+gg}}{\sqrt{bb-gg}}+\frac{a^2\sqrt{bb-gg}}{\sqrt{bb+gg}}=\frac{2\,aa\,bb}{\sqrt{b^4-g^4}},$$

and the difference of the squares of the semi-diameters is

$$\frac{a^2\sqrt{bb+gg}}{\sqrt{bb-gg}}-\frac{a^2\sqrt{bb-gg}}{\sqrt{bb+gg}}=\frac{2\,aa\,gg}{\sqrt{b^4-g^4}}.$$

Now, $\dfrac{2\,aa\,bb}{\sqrt{b^4-g^4}} : \dfrac{2\,aa\,gg}{\sqrt{b^4-g^4}} :: bb : gg.$

Therefore, the area of the external circle is, to the difference between the areas of the alternate circles, as bb is to gg.

And again, because the areas of the circles are, to one another, as the squares of their semi-diameters, and that

$$\left(\frac{ab\sqrt{2}}{\sqrt[4]{b^4-g^4}}\right)^2=\frac{2aa\,bb}{\sqrt{b^4-g^4}}=\frac{a^2\sqrt{bb+gg}}{\sqrt{bb-gg}}+\frac{a^2\sqrt{bb-gg}}{\sqrt{bb+gg}}$$

the area of ABD, the external circle, is equal to the sum of the areas of EFG and HIK, the alternate circles.

But to make the solution possible, $\dfrac{ab\sqrt{2}}{\sqrt[4]{b^4-g^4}}$ must be greater than $\dfrac{a\sqrt[4]{bb+gg}}{\sqrt[4]{bb-gg}}$; in which case b will be greater than g, which may be shown in the following manner.

$$\frac{ab\sqrt{2}}{\sqrt[4]{b^4-g^4}} > \frac{a\sqrt[4]{b^4+g^4}}{\sqrt[4]{bb-gg}}$$

$$\frac{b\sqrt{2}}{\sqrt[4]{b^4-g^4}} > \frac{\sqrt[4]{bb+gg}}{\sqrt[4]{bb-gg}}$$

$$\frac{2bb}{\sqrt{b^4-g^4}} > \frac{\sqrt{bb+gg}}{\sqrt{bb-gg}}$$

$$\frac{2bb}{\sqrt{bb+gg}\sqrt{bb-gg}} > \frac{\sqrt{bb+gg}}{\sqrt{bb-gg}}$$

$$\frac{2bb}{\sqrt{bb+gg}} > \sqrt{bb+gg}$$

$$2bb > bb+gg$$

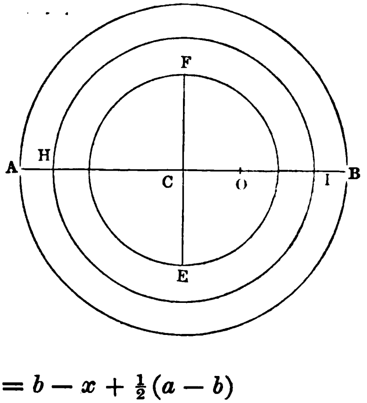
$$bb > gg$$

$$b > g$$

PROPOSITION V.

PROBLEM.—*To divide the diameter of the external circle in such a point, that the rectangle contained by its segments shall be, to that contained by the segments of the diameter of one of the alternate circles, as the diameter of the external circle is to that of the other alternate circle.*

Let AB be a diameter of the external circle, EF the diameter of one alternate circle, and HI that of the other; let C be the common centre; and O the point of section.



Call AB, a; EF, g; HI, b; IO, x; BO, y; and CO, z.

$$\text{Then } a - y = b - x + \tfrac{1}{2}(a - b)$$

$$a - b + x - \tfrac{1}{2}(a - b) = y = x + \frac{a - b}{2}.$$

But $gy(a-y) = ax(b-x)$

$$agy - gyy = ax(b-x)$$

$$gyy - agy = -ax(b-x)$$

$$yy - ay = -\frac{ax(b-x)}{g}$$

$$yy - ay + \frac{aa}{4} = \frac{aa}{4} - \frac{ax(b-x)}{g}$$

$$y - \frac{a}{2} = \pm\sqrt{\frac{aa}{4} - \frac{ax(b-x)}{g}}.$$

$$y = \frac{a}{2} \pm \sqrt{\frac{aa}{4} - \frac{ax(b-x)}{g}}.$$

Now, $x + \frac{a-b}{2} = y = \frac{a}{2} \pm \sqrt{\frac{aa}{4} - \frac{ax(b-x)}{g}}.$

$$x + \frac{a}{2} - \frac{b}{2} = \frac{a}{2} \pm \sqrt{\frac{aa}{4} - \frac{ax(b-x)}{g}}$$

$$x - \frac{b}{2} = \pm\sqrt{\frac{aa}{4} - \frac{ax(b-x)}{g}}$$

$$xx - bx + \frac{bb}{4} = \frac{aa}{4} - \frac{ax(b-x)}{g}$$

$$4xx - 4bx + bb = aa - \frac{4ax(b-x)}{g}$$

$$4gxx - 4bgx + bbg = aag - 4abx + 4axx$$

$$-4axx + 4gxx - 4bgx + 4abx = aag - bbg$$

$$4axx - 4gxz - 4abx + 4bgx = bbg - aag$$

$$xx - bx = \frac{g(bb-aa)}{4(a-g)}$$

$$xx - bx + \frac{bb}{4} = \frac{bb}{4} + \frac{g(bb-aa)}{4(a-g)}$$

$$xx - bx + \frac{bb}{4} = \frac{a(bb-ag)}{4(a-g)}$$

$$x = \frac{b}{2} \pm \sqrt{\frac{a(bb-ag)}{4(a-g)}};$$

Then $y = x + \frac{a-b}{2} = \frac{a}{2} \pm \sqrt{\frac{a\,(bb - ag)}{4\,(a-g)}}$,

And $z = \pm \sqrt{\frac{a\,(bb - ag)}{4\,(a-g)}}$;

Then $zz = \frac{a\,(bb - ag)}{4\,(a-g)}$.

The preceding problem may be otherwise solved in the following manner:—

Call *AH* or *BI*, f; *CA*, t; *CF*, h; *HO*, u; and *IO*, x.

Then, $(f + x) \times (f + u) : ux :: 2t : 2h$.

Also, $2f + x + u = 2t$. Then $x = 2t - 2f - u$;

and $(2t - 2f - x + f)(x + f) : 2tx - 2fx - xx :: t : h$.

Therefore,

$$txx - hxx + 2ftx + htx - 2ttx - 2fhx = ffh - 2fht$$

$$xx + 2fx - 2tx = \frac{fh\,(f - 2t)}{th}.$$

$$xx + 2\,(f - t)\,x = fh\,\frac{(f - 2t)}{t - h};$$

Wherefore, $x = t - f \pm \sqrt{\frac{t\,(tt - 2ft + ff - ht)}{t - h}}$;

And $u = t - f \mp \sqrt{\frac{t\,(tt - 2ft + ff - ht)}{t - h}}$.

Now, $AO = f + u$, and $BO = f + x$;

Therefore, $AO = t \mp \sqrt{\frac{t\,(tt - 2ft + ff - ht)}{t - h}}$,

And $BO = t \pm \sqrt{\frac{t\,(tt - 2ft + ff - ht)}{t - h}}$.

Comparing this last solution with the one preceding it, we find $f = \frac{a-b}{2}$, $h = \frac{g}{2}$, $t = \frac{a}{2}$, and $2ft = \frac{a(a-b)}{2}$.

Now if, in $t - f \pm \sqrt{\frac{t\,(tt - 2ft + ff - ht)}{t - h}}$, the value last found of the unknown quantity x, we substitute for f, h, t, and $2ft$, their values found above, we have

$$x = \frac{b}{2} \pm \sqrt{\frac{\frac{a}{2}\left(\frac{aa}{4} - \frac{a(a-b)}{2} + \frac{(a-b)^2}{4} - \frac{ag}{4}\right)}{\frac{a}{2} - \frac{g}{2}}}$$

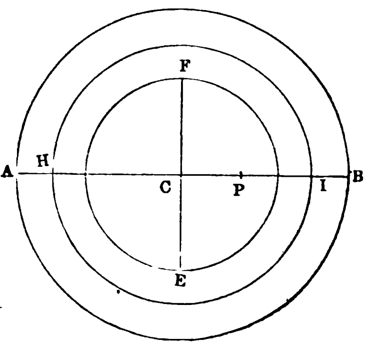
$$= \frac{b}{2} \pm \sqrt{\frac{a\left(\frac{aa}{4} - \frac{a(a-b)}{2} + \frac{(a-b)^2}{4} - \frac{ag}{4}\right)}{a-g}}$$

$$= \frac{b}{2} \pm \sqrt{\frac{a(aa - 2aa + 2ab + aa - 2ab + bb - ag)}{4(a-g)}}$$

$$= \frac{b}{2} \pm \sqrt{\frac{a(bb-ag)}{4(a-g)}},$$ which is the same value as that found in the first solution.

PROPOSITION VI.

Problem.—*To divide the diameter of the external circle in such a point, that the rectangle contained by its segments shall be, to that contained by the segments of the diameter of one of the alternate circles, as the area of the external circle is to that of the other alternate circle.*



Let AB be a diameter of the external circle, EF a diameter of one alternate circle, and HI that of the other. Let C be the common centre, and P the point of section.

Call AB, a; EF, g; HI, b; IP, x; BP, y; and CP, z.

Then, $(a - y)y : (b - x)x :: aa : gg$

And $a - y = b - x + \frac{1}{2}(a - b)$,

$$a - b + x - \tfrac{1}{2}(a - b) = y = x + \frac{a - b}{2};$$

But $ggy(a - y) = aax(b - x)$,

$$yy - ay = \frac{aa}{gg}(xx - bx),$$

$$y = \frac{a}{2} \pm \frac{a}{2g}\sqrt{gg + 4xx - 4bx}.$$

Now $y = x + \frac{a - b}{2}$; therefore

$$x + \frac{a - b}{2} = \frac{a}{2} \pm \frac{a}{2g}\sqrt{gg + 4xx - 4bx},$$

$$xx - bx = \frac{gg(aa - bb)}{4(aa - gg)}$$

$$x = \frac{b}{2} \pm \frac{\sqrt{aabb - 2bbgg + aagg}}{2\sqrt{aa - gg}}.$$

Again—

$$y = x + \frac{a - b}{2} = \frac{a - b}{2} + \frac{b}{2} \pm \frac{\sqrt{aabb - 2bbgg + aagg}}{2\sqrt{aa - gg}}$$

$$y = \frac{a}{2} \pm \frac{\sqrt{aabb - 2bbgg + aagg}}{2\sqrt{aa - gg}};$$

And $z = \frac{a}{2} - y = \pm \frac{\sqrt{aabb - 2bbgg + aagg}}{2\sqrt{aa - gg}}$.

Then, $zz = \frac{aabb - 2bbgg + aagg}{4(aa - gg)}$,

Or, $zz = \frac{bb}{4} + \frac{gg(aa - bb)}{4(aa - gg)}$.

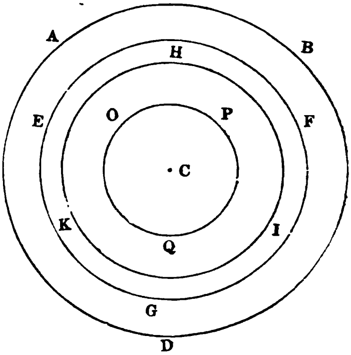
PROPOSITION VII.

PROBLEM.—*Two concentric circles being given, to describe two others, such that their sum shall be equal to the greater, and their difference to the lesser.*

Let the area of the greater circle be aa. Let the area of the lesser be bb. Let x be the semi-diameter of one unknown circle. Let y be the semi-diameter of the other.

Then, as has been shown in the first proposition, cxx is equal to the area of one unknown circle.

And, in like manner, cyy is equal to the area of the other.



Now, $cxx + cyy = aa$; and $cxx - cyy = bb$.

Therefore, $2\,cxx = aa + bb$;

And $2cyy = aa - bb$.

$$\text{Then, } x = \frac{\sqrt{aa + bb}}{\sqrt{2c}};$$

$$\text{And } y = \frac{\sqrt{aa - bb}}{\sqrt{2c}}.$$

Let the greater of the given circles be ABD, and the lesser be OPQ. And let C be the common centre.

Then, from the centre C, at a distance equal to $\frac{\sqrt{aa + bb}}{\sqrt{2c}}$, describe a circle EFG.

From the same centre, at a distance equal to $\frac{\sqrt{aa-bb}}{\sqrt{2c}}$, describe another circle *HIK*. *EFG* and *HIK* shall be the circles required.

The semi-diameter of the greater given circle will be $\frac{a}{\sqrt{c}}$ · The semi-diameter of the lesser given circle will be $\frac{b}{\sqrt{c}}$ ·

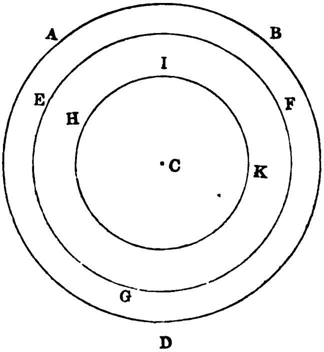
PROPOSITION VIII.

Problem.—*An external circle being given, to describe two alternate circles, the area of one of which shall be a given portion of the area of the external circle.*

Let the area of the external circle be aa.

Let one of the alternate circles be $\frac{aa}{n}$ ·

Let x be the semi-diameter of one alternate circle, and y the semi-diameter of the other.



Then, cxx is the area of one, and cyy of the other.

Now, let $cxx = \frac{aa}{n}$ ·

Then $x \frac{a}{\sqrt{cn}}$ ·

But $cxx + cyy = aa$.

Or $xx + yy = \frac{aa}{c}$ ·

Therefore, $\frac{aa}{cn} + yy = \frac{aa}{c}$ ·

And $yy = \frac{aa}{c} - \frac{aa}{cn}$ ·

Wherefore, $yy = \dfrac{aan - aa}{cn}$,

$$yy = \frac{aa(n-1)}{cn},$$

$$y = \frac{a\sqrt{n-1}}{\sqrt{cn}};$$

The semi-diameter of the external circle will be $\dfrac{a}{\sqrt{c}}$.

Therefore, if, from the centre C, there be described three circles, ABD, EFG, and HIK, with semi-diameters respectively equal to $\dfrac{a}{\sqrt{c}}$, $\dfrac{a\sqrt{n-1}}{\sqrt{cn}}$, and $\dfrac{a}{\sqrt{cn}}$, the two last shall be those required, and the very last is that of which the area is the given portion of the area of the external circle.

PROPOSITION IX.

THEOREM.—*If the semi-diameter of the external circle be cut in extreme and mean ratio by the circumference of the internal circle; the square of the semi-diameter of the alternate circle shall be equal to a rectangle contained by the semi-diameters of the two other circles.*

Let CA, the semi-diameter of the external circle ABD, be cut in extreme and mean ratio in the point E, by the circumference of EFG, the internal circle. Let CH be the semi-diameter of HIK, the alternate circle. Then the square of CH is equal to a rectangle contained by CA and CE.

Call CA, a; CE, b; and CH, x.

Then $AE = a - b$.

Now, because *CA* is cut in extreme and mean ratio, *CA* : *CE* :: *CE* : *AE*;

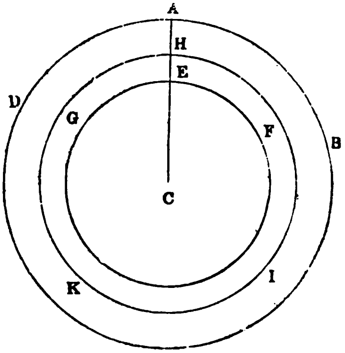
That is,

$a : b :: b : a - b.$

Then,

$bb = aa - ab$, or

$bb + ab = aa.$



Now, because *EFG* and *HIK* are alternate circles, and that circles are, to one another, as the squares of their semi-diameters;

$$xx + bb = aa = bb + ab$$

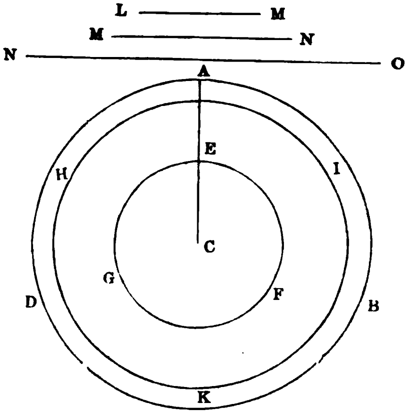
$$xx = ab;$$

That is, the square of *CH*, the semi-diameter of the alternate circle *HIK*, is equal to the rectangle contained by *CA* and *CE*, the semi-diameters of the circles *ABD* and *EFG*.

PROPOSITION X.

THEOREM.—*If three straight lines be harmonical proportionals, the area of one alternate circle be equal to the rectangle contained by the first and second, and the area of the other to a rectangle contained by the second and third, and the third straight line be equal to the double of the second; the semi-diameter of the external circle shall be equal to the double of the semi-diameter of the internal circle.*

Let the lines *LM*, *MN*, and *NO*, be harmonical proportionals; let the area of the alternate circle *EFG* be equal to a rectangle contained by *LM* and *MN*, and let the area of the alternate circle *HIK* be equal to a rectangle contained by *MN* and *NO*; let *CA* be the semi-diameter of the external circle, *CE* the semi-diameter of the internal circle, and *NO* be equal to the double of *MN*; then *CA* shall be equal to the double of *CE*.



Call *LM*, a; *MN*, b; and *NO*, g.

Then, because

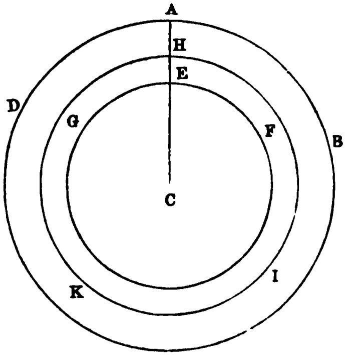
$$g = 2b$$
$$ag = 2ab$$
$$2ag = 4ab$$
$$\sqrt{2ag} = 2\sqrt{ab}$$
$$\frac{\sqrt{2ag}}{\sqrt{c}} = \frac{2\sqrt{ab}}{\sqrt{c}}.$$

Now, by the Scholium to Proposition 2d, $\frac{\sqrt{2ag}}{\sqrt{c}}$ is equal to the semi-diameter of the external circle, and $\frac{\sqrt{ab}}{\sqrt{c}}$ is equal to the semi-diameter of the internal circle. Wherefore, *CA* is equal to the double of *CE*.

PROPOSITION XI.

PROBLEM.—*An external circle being given, to describe two alternate circles, such that the square of the semi-diameter of the greatest shall be equal to the rectangle contained by the semi-diameters of the external and the internal circles.*

Call the semi-diameter of the external circle a; the semi-diameter of the greatest alternate circle y; and that of the internal circle x.



Then,

$$xx + yy = aa, \text{ and}$$

$$ax = yy.$$

Wherefore,

$$xx = aa - yy, \text{ and}$$

$$xx = \frac{y^4}{a^2}$$

$$\frac{y^4}{a^2} = aa - yy$$

$$y^4 = a^4 - a^2y^2$$

$$y^4 + a^2y^2 = a^4$$

$$y^4 + a^2y^2 + \frac{a^4}{4} = \frac{a^4}{4} + a^4 = \frac{5a^4}{4}$$

$$y^2 = \frac{a^2(\sqrt{5} - 1)}{2}$$

$$y = \frac{a\sqrt{\sqrt{5} - 1}}{\sqrt{2}}$$

And, $$x = \frac{yy}{a} = \frac{a^2(\sqrt{5} - 1)}{2a} = \frac{a(\sqrt{5} - 1)}{2}.$$

Then, from C, the centre of the given circle ABD, at distances equal to $\frac{a\sqrt{\sqrt{5}-1}}{\sqrt{2}}$ and $\frac{a(\sqrt{5}-1)}{2}$, describe the circles HIK and EFG.

They shall be those required.

That the squares of the semi-diameters of the two lesser circles are together equal to the square of the semi-diameter of the greatest, is shown by the following calculation :—

$$\frac{a^2(\sqrt{5}-1)}{2} + \frac{a^2(5-2\sqrt{5}+1)}{4} =$$

$$\frac{a^2\sqrt{5}-a^2}{2} + \frac{5a^2-a^2\sqrt{5}+a^2}{4} =$$

$$\frac{4a^2\sqrt{5}-4a^2+10a^2-4a^2\sqrt{5}+2a^2}{8} =$$

$$\frac{-4a^2+10a^2+2a^2}{8} =$$

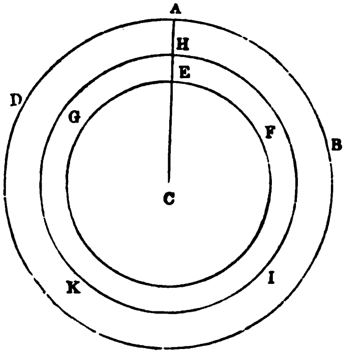
$$\frac{8a^2}{8} = aa.$$

Therefore the lesser circles are alternate.

PROPOSITION XII.

THEOREM.—*If the square of the semi-diameter of the greatest of the two alternate circles be equal to a rectangle contained by the semi-diameters of the two other circles; the semi-diameter of the external circle is cut in extreme and mean ratio by the circumference of the internal circle.*

Let the square of CH, the semi-diameter of the greater alternate circle HIK, be equal to the rectangle contained by *CA* and *CE*, the semi-diameters of the two other circles *ABD* and *EFG*; then *CA* is cut in extreme and mean ratio by *EFG*, the circumference of the internal circle.



Call *CA*, a; *CH*, b; and *CE*, x.

Then $xx + bb = aa$,
and, also, $ax = bb$.

Wherefore,

$$xx + ax = aa,$$

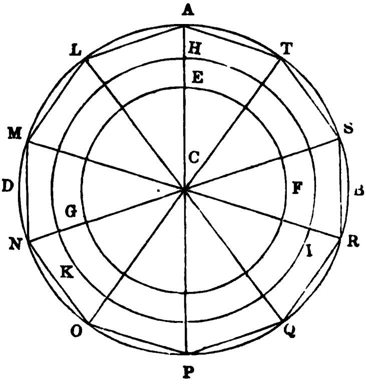
or, $xx = aa - ax$.

Therefore, $a : x :: x : a - x$;
That is, $CA : CE :: CE : EA$.

PROPOSITION XIII.

Problem.—*To describe an equilateral and equiangular decagon in a given circle.*

Within the given circle *ABD*, describe two alternate circles, *EFG*, and *HIK*, such that the square of *CH*, the semi-diameter of *HIK*, the greatest, shall be equal to the rectangle contained by *CA* and *CE*, the semi-diameters of the external and the internal circles.



In the given circle AB, place ten straight lines, AL, LM, MN, &c., adjacent to one another, and each equal to CE, the semi-diameter of the internal circle.

Join CL, CM, CN, CO, &c.

Call a right angle, r.

Then, by the 12th Proposition of this Book, $\overline{CE}^2 = CA \times EA$.

Therefore, by the 10th Proposition of Book IV. ELEMENTS, each of the angles, CAL, CLA, at the base of the isosceles triangle ACL, is double of the angle ACL, which is at the vertex.

But the sum of these three angles is $2r$.

Therefore, the sum of the two angles at the base is $2r - \frac{2r}{5} = \frac{10r - 2r}{5} = \frac{8r}{5}$. And these angles are equal to one another, and each is equal to the adjacent angle of the adjacent triangle.

Now, one interior angle of the polygon, $ALMNO$, &c., is equal to two angles of the triangle, and consists of two such adjacent angles.

Therefore, one interior angle of the polygon is equal to $\frac{8r}{5}$. And these interior angles of the polygon are equal to one another.

And the sum of ten such angles is $\frac{80r}{5} = 16r$.

And, by the 1st Corollary to the 32d Proposition of Book I. ELEMENTS, all the interior angles of a decagon must be equal to $20r - 4r = 16r$.

And that would not be the case, if the rectilineal figure had more or fewer sides.

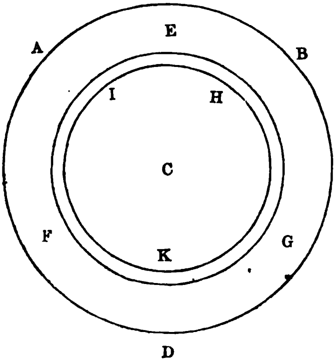
Therefore, the figure is a decagon; and all its sides are equal.

Therefore, in the circle ABD, there has been described an equiangular and equilateral decagon; which was to be done.

PROPOSITION XIV.

Problem.—*To describe two alternate circles, such that the internal circle shall be to the other as that other to a third circle, and that the sum of the two shall be equal to a fourth.*

Let the semi-diameter of the third circle be a, and let bb be equal to the squares of the semi-diameters of the two alternate circles.



Call the semi-diameter of the internal circle x, and the semi-diameter of the other, y.

Then,

$x^2 : y^2 :: y^2 :: a^2$,

and, $x^2 + y^2 = b^2$.

Wherefore, $y^4 = a^2x^2$,

and $y^2 = b^2 - x^2$

$y^2 = ax = b^2 - x^2$

$$x^2 + ax = b^2$$

$$x^2 + ax + \frac{a^2}{4} = \frac{a^2 + 4b^2}{4}$$

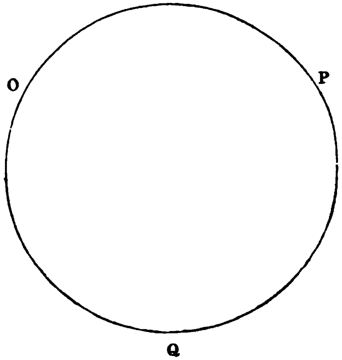
$$x = \pm \frac{\sqrt{a^2 + 4b^2} - a}{2}.$$

Then, because $y^2 = ax = \dfrac{\pm a\sqrt{a^2 + 4b^2} - a^2}{2}$

$$y = \pm \sqrt{\frac{\pm a\sqrt{a^2 + 4b^2} - a^2}{2}}.$$

Let the circle, *OPQ*, have a semi-diameter equal to a.

From a centre, *C*, with a radius equal to *b*, describe the circle ***ABD***.



From the same centre, with a radius equal to

$$\pm\frac{\sqrt{a_2 + 4b^2} - a}{2},$$

describe the circle, ***IHK***.

From the same centre, with a radius equal to

$$\pm\sqrt{\frac{\pm a\sqrt{a^2 + 4b^2} - a^2}{2}},$$ describe the circle ***EFG***.

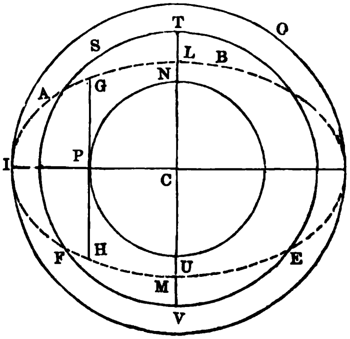
The two last circles shall be those required.

PROPOSITION XV.

Problem.—*An ellipse being given, to describe two alternate circles, and draw an ordinate to the axis, so that it shall be equal to the diameter of the circle of which it is the tangent.*

Let the given ellipse be ***ABEF***.

Required, two alternate circles, ***STV***, and ***NPU***, and an ordinate, ***GH***, being a tangent to ***NPU***, and such that the diameter ***NU*** shall be equal to ***GH***.



By the 1st Proposition of the Third Book of the Treatise, find *C*, the centre of the ellipse; and, by the 33d Proposition of the same Book, find *IK* and *LM*, the axes.

Call *IK*, a; *LM*, b; *GH*, y; and *TV*, u.

Then, because $TV : GH :: IK : LM$,

$$u : y :: a : b,$$

$$\text{and } GH = y = \frac{bu}{a};$$

$$\text{Wherefore, } NU = \frac{bu}{a}.$$

From *C*, the centre of the ellipse, with a radius equal to *CI*, describe an external circle, *IOK*.

Then, because $\overline{NU}^2 + \overline{TV}^2 = \overline{IK}^2$

$$\frac{b^2u^2}{a^2} + u^2 = a^2.$$

Wherefore, $$(a^2 + b^2)u^2 = a^4,$$

$$u^2 = \frac{a^4}{a^2 + b^2},$$

and $$u = \frac{a^2}{\sqrt{a^2 + b^2}}.$$

Then, from the centre *C*, with a radius equal to $\frac{aa}{2\sqrt{aa + bb}}$, describe a circle *STV*.

From the same centre, with a radius equal to $\frac{ab}{2\sqrt{aa + bb}}$, describe a circle *NPU*.

And through *P*, the point in which the last circle cuts *IK*, the axis, draw *GH* at right angles to *IK*, and meeting the circumference of the ellipse in the points *G* and *H*.

The circles *STV* and *NPU* shall be those required, and *GH* shall be the ordinate equal to the diameter *NU*.

For, because $CN = \frac{ab}{2\sqrt{aa + bb}}$,

$$NU = 2CN = \frac{ab}{\sqrt{aa+bb}}.$$

Now, $GH = \frac{bu}{a}$, and $\overline{GH}^2 = \frac{b^2u^2}{a^2} =$

$$\frac{bb \times \frac{aaaa}{aa+bb}}{aa} = \frac{aaaabb}{aaaa+aabb} = \frac{a^2b^2}{a^2+b^2}.$$

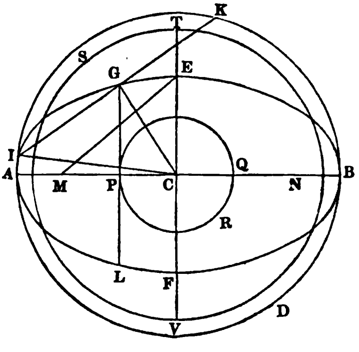
Wherefore, $GH = \frac{ab}{\sqrt{a^2+b^2}} = NU.$

PROPOSITION XVI.

THEOREM.—*If the greater axis of an ellipse be divided by an ordinate, the absciss be the radius of a circle, and a chord to the external circle be drawn through the end of the ordinate, at right angles to a line drawn from the same end to the centre; the diameter of the alternate circle shall not be less than the chord.*

Let *AB*, the greater axis of the ellipse *AEBF*, be divided at the point *P*, by the ordinate *GL*.

Let *CP* be the radius of the circle *PQR*, and let *STV* be the alternate circle. Let *CG* be joined. Through *G*, at right angles to *CG*, let there be drawn *IK*, a chord to the external circle *ABD*, and let *M* and *N* be the focuses.



Then *TV* is not less than *IK*.

For, by the 25th Proposition of the Third Book of the Treatise, $AB : MN :: TV : IK$.

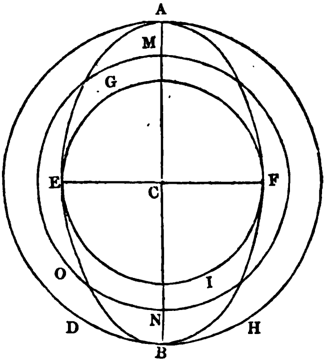
Now, by the 12th and 13th Definitions of the same Book, AB is not less than MN.

Therefore, TV is not less than IK.

PROPOSITION XVII.

THEOREM.—*The greater axis remaining the same, the greater that the square of the lesser axis is, the lesser is the square of the line that joins the focuses.*

Let AB be the greater axis of the ellipse $AEBF$, EF the lesser axis, and MN the line that joins the focuses. Then, AB remaining the same, the greater that the square of EF is, the less is the square of MN.



From C, the centre of the ellipse, at the distance CA, describe the circle ABD; and from the same centre describe the circles EFG and MNO.

Then, by the 12th and 13th Definitions of the Third Book of the Treatise, ABD is the external circle, and EFG and MNO are alternate circles.

And, because circles are to one another as the squares of their diameters, $EFG : MNO :: \overline{EF}^2 : \overline{MN}^2$.

And, by the 14th Proposition of the First Book of the Treatise, AB remaining the same, the greater that EFG is, the less is MNO.

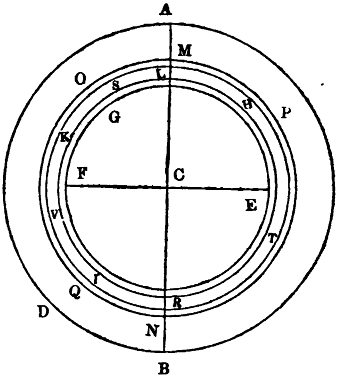
Therefore, the greater that the square of EF is, the less is the square of MN.

PROPOSITION XVIII.

THEOREM.—*If the external circle have, for its diameter, a diameter of the ellipse, and another circle cut that diameter in the principal points, and an alternate circle be described; the conjugated diameter of the ellipse shall be equal to a diameter of the alternate circle.*

Let the external circle *ABD*, have for its diameter *AB*, a diameter of the ellipse; let another circle, *OPQ*, cut *AB* in *M* and *N*, the principal points; and let an alternate circle, *EFG*, be described; the conjugated diameter of the ellipse shall be equal to a diameter of *EFG*.

For, if not, from *C*, the common centre, with a radius equal to half the conjugated diameter, describe another circle, *HIK*. Also describe a circle, *STV*, alternate with *HIK*.



Now, because, by the 14th Proposition of the First Book of the Treatise, the greater that the area of one alternate circle is, the less is that of the other; if *HIK* be greater than *EFG*, *STV* must be less than *OPQ*.

The circumference of *STV* must, therefore, cut the diameter *AB*, at some points, as *L* and *R*, between the centre *C*, and the principal points *M* and *N*; and, consequently, by the 12th Definition of the Third Book of the Treatise, *L* and *R* are the principal points.

But, by the conditions of the Proposition, *M* and *N* are the principal points.

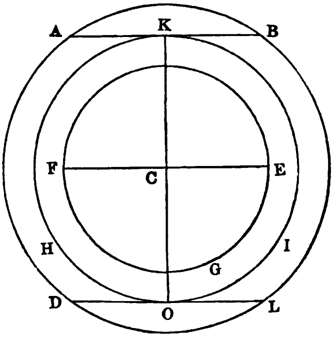
Then, the straight line *MN* is equal to the straight line *LR*, the greater to the less, which is impossible.

In like manner may be demonstrated that, if *HIK* be less than *EFG*, *STV* must be greater than *OPQ*; and that the circumference of *STV* must therefore cut *AB* in one point between *A* and *M*, and in another point between *B* and *N*. And hence will follow a similar impossibility.

PROPOSITION XIX.

THEOREM.—*If, in an external circle, there be placed a chord equal to the diameter of one alternate circle, that chord will be a tangent to the other alternate circle.*

In the external circle *ABD*, let there be placed a chord *AB*, equal to *EF*, the diameter of the alternate circle, *EFG*. Let the other alternate circle be *HIK*.



AB will be a tangent to *HIK*.

From the point *D*, draw *DO*, a tangent to *HIK*, and produce it to *L*, a point in the circumference of *ABD*.

From the point *C*, the centre of the circles, let fall on *AB* and *DL*, the perpendiculars *CK* and *CO*.

Now, because *DL* is a tangent to *HIK*, *DL* is equal to *EF*, by the 6th Proposition of the Third Book of the Treatise.

And, by the conditions of this Proposition, *AB* is equal to *EF*.

Wherefore, *AB* is equal to *DL*.

Therefore, by the 14th Proposition of the Third Book of ELEMENTS, AB and DL are equally distant from the centre C.

Wherefore, by the 3d Definition of the same Book, the perpendiculars CK and CO are equal.

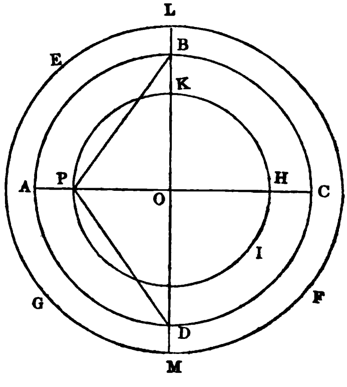
Therefore, by the 11th Definition of the First Book of ELEMENTS, the points K and O are in the circumference of the circle.

Wherefore, by the Corollary to the 16th Proposition of the Third Book of Elements, AB is a tangent to HIK.

PROPOSITION XX.

THEOREM.—*The rectangle contained by two straight lines, drawn from the point in which the circumference of the internal circle is cut by the diameter of the alternate circle, to the points in which the circumference of the alternate circle is cut by the diameter of the external circle, drawn at right angles to the other diameter, is equal to the sum of the squares of the radiuses of the alternate circles.*

Let EFG be a circle, external to the alternate circles ABC and HIK, of which the latter is the internal circle. Let AC, a diameter of ABC, cut the circumference HIK in the point P. Let LM, a diameter of the external circle, and at right angles to AC, cut the circumference ABC in the points B and D. Let PB and PD be joined, and let O be the common centre of the three circles.



Then, $PB \times PD = \overline{OA}^2 + \overline{OP}^2$.

For, by the 1st Corollary to the 4th Proposition of the Treatise on the ELLIPSE, $OL = PB = PD$.

And, by the 47th Proposition of the First Book of ELEMENTS,

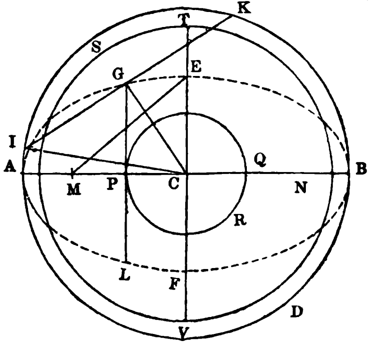
$$\overline{PB}^2 = \overline{OB}^2 + \overline{OP}^2 = \overline{OA}^2 + \overline{OP}^2 = PB \times PD.$$

COR.—Because the circumference of the circle $ABCD$, of which the centre is O, has been divided into four equal parts, AB, BC, CD, DA, and that, from each point of division, A, B, C, D, there has been drawn to P, a point in the radius of that circle, a straight line, AP, BP, CP, DP, and that the rectangle contained by BP and DP, the lines drawn from B and D, the alternate points of division, is equal to the sum of the squares of OA and OP; this result is a simple case of Cotes's theorem.

PROPOSITION XXI.

THEOREM.—*If the greater axis of an ellipse be divided by an ordinate, the absciss be the radius of a circle, and a chord to the external circle be drawn through the end of the ordinate, at right angles to a line drawn from the same end to the centre; the diameter of the alternate circle shall be to the chord, as the greater axis is to the line which joins the focuses.*

Let AB, the greater axis of the ellipse $AEBF$, be divided, at the point P, by the ordinate GL. Let CP be the radius of the circle PQR, and let STV be the alternate circle. Let CG be joined. Through



G, at right angles to CG, let there be drawn IK, a chord to the external circle ABD; and let M and N be focuses.

Then $TV : IK :: AB : MN$.

Join CI and EM.

Then, because STV and PQR are alternate circles, the area of STV is equal to the difference of the areas of ABD and PQR.

Then, $\overline{CT}^2 = \overline{AC}^2 - \overline{CP}^2$.

Now, by the 15th Proposition of the Treatise on the Ellipse, $TV : GL :: AB : EF$.

Then $\frac{1}{2}TV : \frac{1}{2}GL :: \frac{1}{2}AB : \frac{1}{2}EF$.

That is, $CT : GP :: AC : EC$.

And $\overline{CT}^2 : \overline{GP}^2 :: \overline{AC}^2 : \overline{EC}^2$.

Now, $CT^2 = AC^2 - CP^2$.

And $\overline{GP}^2 = \overline{CG}^2 - \overline{CP}^2$.

Therefore,

$$\overline{AC}^2 - \overline{CP}^2 : \overline{CG}^2 - \overline{CP}^2 :: \overline{AC}^2 : \overline{EC}^2.$$

Alternately,

$$\overline{AC}^2 - \overline{CP}^2 : \overline{AC}^2 :: \overline{CG}^2 - \overline{CP}^2 : \overline{EC}^2.$$

Therefore, by the Corollary to the 19th Proposition of Book V. Elements, $\overline{AC}^2 - \overline{CP}^2 - \overline{CG}^2 + \overline{CP}^2 : \overline{AC}^2 - \overline{EC}^2 :: \overline{CG}^2 - \overline{CP}^2 : \overline{EC}^2$.

That is,

$$\overline{AC}^2 - \overline{CG}^2 : \overline{AC}^2 - \overline{EC}^2 :: \overline{CG}^2 - \overline{CP}^2 : \overline{EC}^2.$$

Now, $CI = CA$; $\overline{CI}^2 = \overline{CA}^2$;

$$\overline{CA}^2 - \overline{CG}^2 = \overline{CI}^2 - \overline{CG}^2 = \overline{IG}^2.$$

And, by the 24th Proposition of the Treatise,

$$EM = CA.$$

Therefore, $\overline{EM}^2 = \overline{CA}^2$, and $\overline{CA}^2 - \overline{CE}^2 = \overline{EM}^2 - \overline{CE}^2 = \overline{CM}^2$.

Then, because $\overline{CG}^2 - \overline{CP}^2 = \overline{GP}^2$;

$$\overline{IG}^2 : \overline{CM}^2 :: \overline{GP}^2 : \overline{CE}^2.$$

Now, because $\overline{CT}^2 : \overline{GP}^2 :: \overline{CA}^2 : \overline{CE}^2$;

or $\overline{CT}^2 : \overline{CA}^2 :: \overline{GP}^2 : \overline{CE}^2$; $\overline{IG}^2 : \overline{CM}^2 :: \overline{CT}^2 : \overline{CA}^2$.

Then, $IG : CM :: CT : CA$;

or, $IG : CT :: CM : CA$;

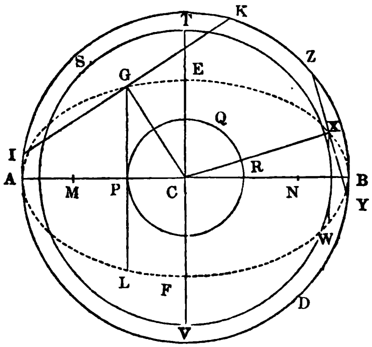
and $CT : IG :: CA : CM$.

Therefore, $2CT : 2IG :: 2CA : 2CM$;

or, $TV : IK :: AB : MN$.

COR. I.—If, with the same regard to the other alternate circle, another ordinate and another chord be drawn; the two chords shall be to one another as the diameters of the alternate circles.

Through the point in which STV divides the greater axis, let there be drawn WX, another ordinate to the ellipse.



Let CX be joined, and, through the point X, and at right angles to CX, let there be drawn YZ, a chord to the external circle.

Then, IK is to YZ as TV to PR.

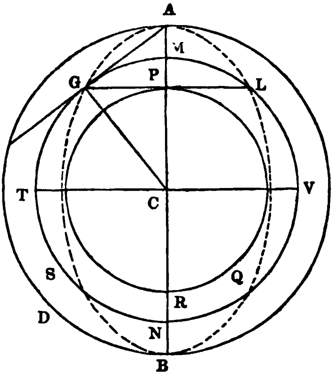
For, by the foregoing Proposition,

$$TV : IK :: AB : MN; \text{ and, also,}$$

$$PR : YZ :: AB : MN.$$

Wherefore, $TV : IK :: PR : YZ$; and, alternately, $TV : PR :: IK : YZ$.

Cor. ii.—If one of the circles cut the axis at a focus, its diameter is a mean proportional between the axis, and the chord drawn through the end of the ordinate, cutting the axis at the same point as the other alternate circle.



For, because

$$AB : MN :: TV : IK;$$

and now,

$$MN = TV;$$

$$AB : TV :: TV : IK.$$

Cor. iii.—If the extremity of the chord, drawn through the extremity of the ordinate cutting the axis in the point in which it is cut by one circle, meet the extremity of the axis; the diameter of the alternate circle shall be a mean proportional between the axis and the diameter of the other.

For, by the conditions of the Proposition, the angle at P is a right angle, as is also the angle CGA; therefore, by the Corollary to the 8th Proposition of Book VI. Elements,

$$AC : CG :: CG : CP;$$

therefore,

$$2AC : 2CG :: 2CG : 2CP;$$

that is,

$$AB : TV :: TV : PR.$$

Cor. IV.—If one of the circles cut the axis at a focus, and the extremity of the chord, drawn through the extremity of the ordinate cutting the axis in the point in which it is cut by the alternate circle, meet the extremity of the axis; the chord shall be equal to the diameter of the internal circle.

For, by the present Proposition,

$$TV : IK :: AB : MN;$$

that is, in the present case,

$$TV : IA :: AB : MN; \text{ or,}$$

$$AB : MN :: TV : IA;$$

that is, as the circle cuts the axis at a focus,

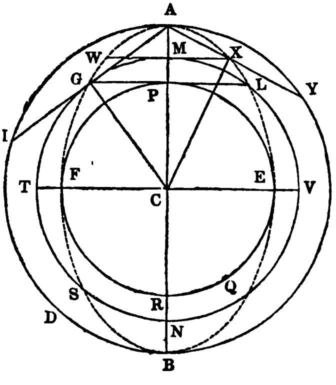
$$AB : TV :: TV : IA.$$

Now, by the 3d Corollary,

$$AB : TV :: TV : PR.$$

Therefore, by the 9th Proposition of the Fifth Book of Elements, $IA = PR$.

Cor. V.—If, as in the 1st Corollary, two chords be drawn, and their extremities meet the extremity of the axis; that which is drawn through the extremity of the ordinate, cutting the axis in the point in which it is cut by the internal circle, shall be a mean proportional between the other chord, and the diameter of the alternate circle.



For, by the 1st Corollary, $TV : PR :: IK : YZ$.

And, in the present case, IK becomes IA, and YZ becomes YA.

Therefore, $TV : P\dot{R} :: IA : YA$.

Now, by the 4th Corollary, $PR = IA$.

Therefore, $TV : IA :: IA : YA$.

COR. VI.—If the greater alternate circle cut the axis at a focus, and the extremity of the chord, drawn through the extremity of the ordinate cutting the axis in the point in which it is cut by the internal circle, meet the extremity of the axis; the chord will be a tangent to the alternate circle.

For, by the 4th Corollary, the chord is then equal to the diameter of the internal circle.

Wherefore, by the 19th Proposition, the chord is a tangent to the alternate circle.

COR. VII.—If the greater alternate circle cut the axis at a focus, and the extremity of the chord, drawn through the extremity of the ordinate cutting the axis in the point in which it is cut by the internal circle, meet the extremity of the axis; the rectangle contained by the lines that join each focus, and the extremity of the chord, shall be equal to the square of half the chord.

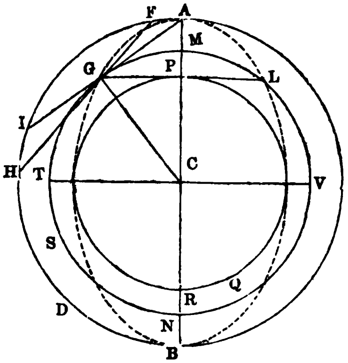
For, by the conditions of the Proposition, CG is at right angles to IA. Therefore, by the 3d Proposition of the Third Book of ELEMENTS, AG is the half of IA.

Now, from A, a point without the circle STV, there are drawn two straight lines, AN and AG; of which AN cuts the circle, and AG touches it. Therefore, by the 36th Proposition of the Third Book of ELEMENTS, the rectangle contained by AN and AM is equal to the square of AG.

COR. VIII.—If the greater alternate circle cut the axis at a focus, the chord, drawn through the extremity of the ordinate cutting the axis in the point in which it is cut by the internal circle, shall meet the extremity of the axis.

Let *A* be the extremity of the axis, and *G* the extremity of the ordinate, cutting the axis in the point in which it is cut by the internal circle. Join *AG*, and produce it to *I*, a point in the circumference.

Then, if it be possible, let the chord meet the circumference of the external circle in a different point, as at *F*. Join *GF*.



Now, by the 14th Proposition of the First Book of Elements, *GF* is not in the same straight line with *GI*.

Then, produce *FG* to *H*, a point in the circumference. By construction, *CG* is perpendicular to *HF*.

Now, by the 6th Corollary, *IA*, meeting the extremity of the axis, is a tangent to the alternate circle *STV*. Therefore, by the 18th Proposition of the Third Book of Elements, *CG* is perpendicular to *IA*. Wherefore, the angles, *CGA*, *CGF*, are equal, the less to the greater, which is absurd.

Therefore, the chord does not meet the circumference of the external circle at the point *F*.

In the same manner, the chord may be demonstrated to meet the circumference in no other points but *I* and *A*, of which one is the extremity of the axis.

Cor. ix.—If the circumference of the greater alternate circle cut the axis at a focus, its diameter shall be, to the lesser axis, as the chord to the ordinate drawn through the point in which the circumference of the lesser cuts the axis.

For, by the 15th Proposition of the Third Book of the Treatise,

$$TV : GL :: AB : EF.$$

And, by the present Proposition,

$$TV : IA :: AB : MN.$$

Now, in the present case, $TV = MN$.

Therefore, $MN : GL :: AB : EF$;

And, $MN : IA :: AB : MN$.

Therefore, $\frac{MN \times EF}{GL} = AB = \frac{\overline{MN}^2}{IA}$.

Then, $MN \times GL = EF \times IA$.

And, $MN : EF :: IA : GL$.

Cor. x.—In the same case, the diameter of the greater alternate circle shall be to that of the lesser, as the chord to the same ordinate.

For, by the 16th Proposition of the Third Book of the Treatise, the lesser axis is now the diameter of the lesser alternate circle.

Cor. xi.—If the circumference of the greater alternate circle cut the axis at a focus, the diameter of the lesser shall be a mean proportional between the axis and the ordinate drawn through the focus.

For, by the 15th Proposition of the Third Book of the Treatise,

$$PR : WX :: AB : EF.$$

And, by the 16th Proposition of the same Book,

$$EF = PR.$$

Therefore, $PR : WX :: AB : PR$.

And, $AB : PR :: PR : WX$.

Cor. xii.—If the greater alternate circle cut the axis at a focus, and, as in the 1st Corollary, two chords be drawn, and their extremities meet the extremity of the axis; the chord drawn through the extremity of the ordinate drawn through the focus, shall be a mean proportional between the ordinate and the diameter of the internal circle.

For, by the 5th Corollary,

$$TV : IA :: IA : YA.$$

And, by the 4th Corollary,

$$PR = IA.$$

Therefore, $TV : PR :: PR : YA.$

Now, by the 11th Corollary,

$$AB : PR :: PR : WX.$$

Then, $\overline{PR}^2 = TV \times YA;$

And also, $\overline{PR}^2 = AB \times WX.$

Therefore, $AB \times WX = TV \times YA.$

Then, $AB : TV :: YA : WX.$

Now, $TV = MN.$

Therefore, $AB : MN :: YA : WX.$

And because, by the 5th Definition of the First Book of the Treatise, the internal circle is now the alternate circle; by the present 21st Proposition,

$$AB : MN :: PR : YA.$$

Therefore, by the 11th Proposition of the Fifth Book of Elements,

$$PR : YA :: YA : WX.$$

PROPOSITION XXII.

THEOREM.—*If a diameter of an ellipse be parallel to one of the sides of a parallelogram inscribed in the ellipse, it cannot be also a diameter of the parallelogram.*

For, by the Definition subjoined to the 34th Proposition of the First Book of ELEMENTS, a diameter of a parallelogram joins its opposite angles.

Therefore, if this diameter were parallel to one of the sides of the parallelogram, two parallel straight lines would meet; which is absurd.

COR.—A diameter of an ellipse, if parallel to one of the sides of a parallelogram inscribed in the ellipse, must cut two other sides of the parallelogram.

PROPOSITION XXIII.

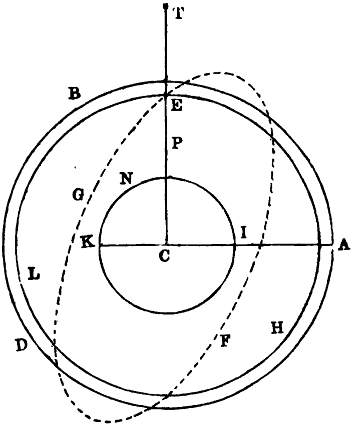
PROBLEM.—*A semi-diameter of an ellipse being given; to make it a mean proportional between an absciss taken on it, and the given semi-diameter produced.*

Let CE be a semi-diameter of the ellipse EFG.

CE is required to be made a mean proportional between some absciss, as CP, and the semi-diameter produced, as to T.

From the centre C, at the distance CE, describe the circle LEH.

At some distance greater or less than CE describe another circle, NIK.



To these, as alternate circles, describe an external circle *ABD*.

Through the centre *C*, draw the straight line *AIK*, cutting the circumference of the alternate circle *NIK*, in the points *I* and *K*.

Upon the line *CE*, take *CP*, equal to *AI*, and produce *CE* to *T*, making *CT* equal to *AK*.

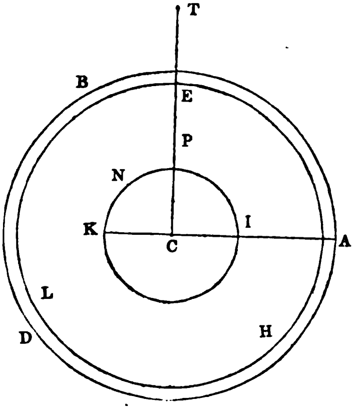
Then, *CE* will be a mean proportional between *CP* and *CT*.

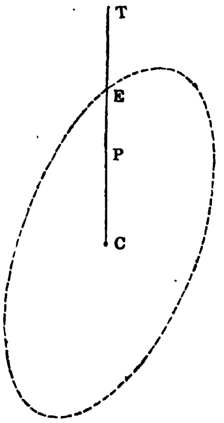
For, by the 11th Proposition of the First Book of the Treatise, *CE* is a mean proportional between *AI* and *AK*.

Now, *CP* is equal to *AI*, and *CT* is equal to *AK*.

Therefore, by the 7th Proposition of the Fifth Book of ELEMENTS, $CP : CE :: CE : CT$.

SCHOL.—The circles, determining the proportions of the parts of the semi-diameter produced, need not be concentric with the ellipse, but may be quite detached from it. For the 11th Proposition of the First Book of the Treatise shows a concise way of obtaining, without calculation, and without scale, three lines, of which two are arbitrary, and of which one is a mean proportional.

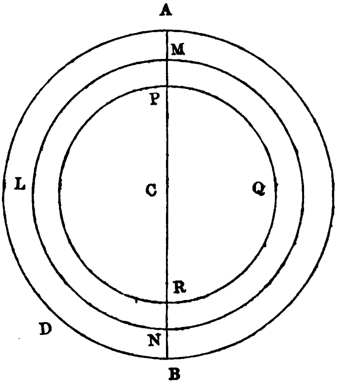




PROPOSITION XXIV.

THEOREM.—*If the diameter of an external circle be cut, on the same side of the centre, by the circumferences of two alternate circles; the radius of one of the alternate circles shall be a mean proportional between the parts into which the diameter of the external circle is divided by the circumference of the other.*

Let AB, the diameter of the external circle ABD, be cut in the points M and P, on the same side of C, the centre, by the circumferences of the alternate circles LMN and PQR; then PC, the radius of one circle, shall be a mean proportional between AM and MB, the parts into which AB is divided by the circumference LMN.



For, by the 2d Defininition of the First Book of the Treatise,

$$\overline{MN}^2 + \overline{PR}^2 = \overline{AB}^2.$$

Now,

$$\overline{PR}^2 = 4\overline{PC}^2, \text{ and } MN = AB - 2AM.$$

Then,

$$\overline{MN}^2 = \overline{AB}^2 - 4AB \times AM + 4\overline{AM}^2.$$

Therefore,

$$\overline{AB}^2 - 4AB \times AM + 4\overline{AM}^2 + 4\overline{PC}^2 = \overline{AB}^2.$$

Wherefore,

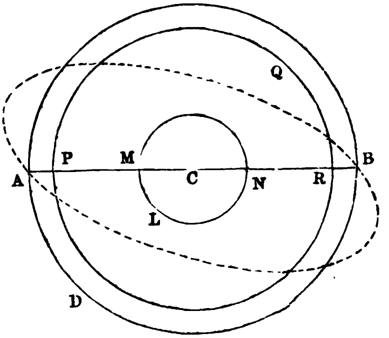
$$-4AB \times AM + 4\overline{AM}^2 + 4\overline{PC}^2 = 0.$$

Then,

$$4\overline{PC}^2 = 4AB \times AM - 4\overline{AM}^2.$$

Or, $\overline{PC}^2 = AB \times AM - \overline{AM}^2.$
Therefore, $AM : PC :: PC : AB - AM,$
Or, $AM : PC :: PC : MB.$

COR.—If a diameter common to an ellipse and an external circle be divided in the same manner by two alternate circles, the same parts will bear the same proportions to the radius.



PROPOSITION XXV.

PROBLEM.—*A diameter common to an ellipse and to a circle being given, to take on it an absciss, such that the corresponding ordinate shall be equal to the given diameter; and to make the circle external to two alternate circles, of one of which the absciss shall be a radius.*

Let AB be a diameter common to the ellipse $AEBF$, and to the circle ABD. There is required, upon it, CP, an absciss, such that GH, the corresponding ordinate, shall be equal to AB, and that the circle ABD shall be external to two alternate circles, PQX and STV; CP being a radius of PQX.

Draw EF, the conjugated diameter.

Suppose the alternate circles described, and let PQ be a diameter of the circle PQX, and TV a diameter of the alternate circle STV.

Call AB, a; EF, b; PQ, x; and TV, y.

Then, by the 15th Proposition of the Third Book of the Treatise,

$$TV : GH :: AB : EF;$$

that is, $y : a :: a : b$; or, $y = \frac{aa}{b}$,

and, $$yy = \frac{aaaa}{bb}.$$

Again, $$aa = xx + yy,$$

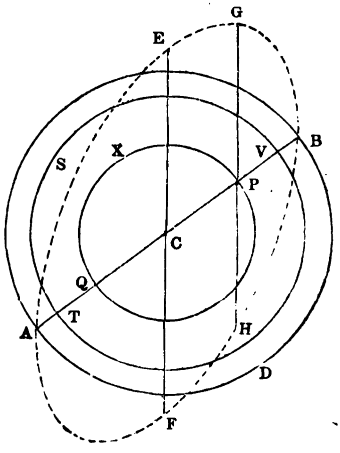
$$yy = aa - xx = \frac{aaaa}{bb}.$$

Then, $$-xx = \frac{aaaa}{bb} - aa,$$

$$xx = aa - \frac{aaaa}{bb},$$

$$xx = \frac{aabb - aaaa}{bb} = \frac{aa}{bb}(bb - aa),$$

$$x = \frac{a}{b}\sqrt{bb - aa}.$$



Therefore, from the centre C, at the distance CP, equal to

$$\frac{a}{2b}\sqrt{bb - aa},$$

describe the circle PQX; and, by the 4th Proposition of the First Book of the Treatise, describe the alternate circle STV.

Through P, the point in which the diameter AB is cut by the circumference

PQX, draw the ordinate *GH*. It shall be that required.

Cor.—The equation $x = \frac{a}{b}\sqrt{bb - aa}$ shows that if *AB* be greater than *EF*, the problem is impossible.

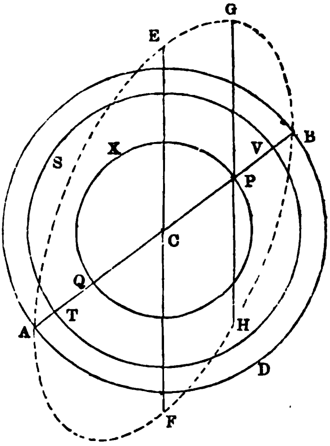
PROPOSITION XXVI.

Theorem.—*If the absciss of an ellipse be the radius of a circle, and the corresponding ordinate be equal to the diameter common to the ellipse and an external circle; the diameter of the alternate circle shall be to the ordinate as the ordinate to the conjugate.*

For, by the 15th Proposition of the Third Book of the Treatise, the diameter of the alternate circle shall be to the ordinate as the diameter of the ellipse to its conjugate.

Now, here, the ordinate is equal to the diameter of the ellipse.

Therefore, by the 7th Proposition of the Fifth Book of Elements, the diameter of the alternate circle shall be to the ordinate as the ordinate to the conjugate.

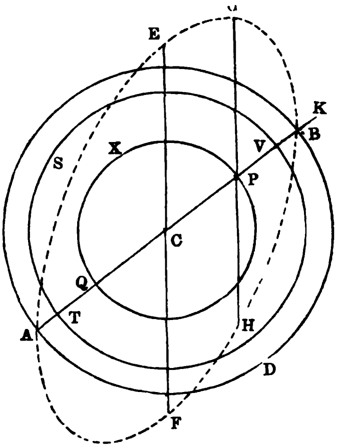


OTHERWISE.

Let *CP*, the absciss of the ellipse *AEBF*, be the radius of the circle *PQX*, and let *GH*, the corresponding ordinate, be equal to *AB*, a diameter common to the ellipse $AEBF$, and the external circle ABD; let EF be the conjugated diameter, and let TV be the diameter of the alternate circle STV; then TV shall be to GH as GH to EF.

For, by the 15th Proposition of the Third Book of the Treatise, TV is to GH as AB to EF.

Now, in the present case, GH is equal to AB.



Therefore, by the 7th Proposition of the Fifth Book of ELEMENTS, TV is to GH as GH to EF.

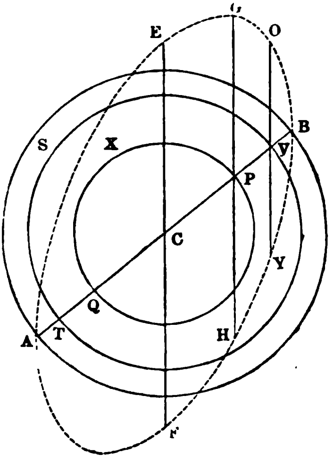
COR.—If CP be produced so that CK shall be equal to CE; CB, the semi-diameter of the ellipse, will be a mean proportional between the absciss taken on it and the semi-diameter produced.

For, because TV is to GH as GH to EF; and CV is the half of TV, CB equal to the half of GH, and CK equal to the half of EF; CV is to CB as CB to CK.

PROPOSITION XXVII.

THEOREM.—*If the absciss of an ellipse be the radius of a circle, and the corresponding ordinate be equal to the diameter common to the ellipse and an external circle, and, through the point in which the circumference of the alternate circle cuts the common diameter, there be drawn an ordinate to the ellipse; the semi-ordinates shall be to one another, alternately, as the semi-diameters of the alternate circles.*

Let CP, the absciss of the ellipse $AEBF$, be the radius of the circle PQX, and let GH, the corresponding ordinate, be equal to AB, a diameter common to the ellipse $AEBF$, and the external circle ABD; let EF be the conjugated diameter, let TV be the diameter of the alternate circle STV, and through the point V, let the ordinate OY be drawn.



Then,

$PH : OV :: CV : CP.$

For, by the 15th Proposition of the Third Book of the Treatise,

$TV : GH :: AB : EF.$

And, by the same Proposition,

$$PQ : OY :: AB : EF.$$

Therefore, $TV : GH :: PQ : OY,$

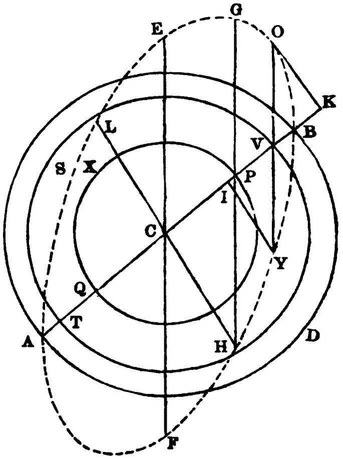
And, $\tfrac{1}{2}TV : \tfrac{1}{2}GH :: \tfrac{1}{2}PQ : \tfrac{1}{2}OY.$

Or, $CV : PH :: CP : OV,$

$CV : CP :: PH : OV,$

$PH : OV :: CV : CP.$

Cor. i.—If half the common diameter be produced, so that it shall be equal to half the conjugate, and a straight line be drawn from the end of the produced semi-diameter to the end of the ordinate drawn through the point in which that diameter is cut by the circumference of the alternate circle; that straight line shall be parallel to a diameter drawn from the alternate end of the other ordinate.



From the point H, draw the diameter HL. Produce the semi-diameter CB to K, making CK equal to CE. Join KO.

KO and HL shall be parallel.

For, because circles are, to one another, as the squares of their semi-diameters, and that ABD is external to the alternate circles STV and PQX,

$$\overline{CV}^2 + \overline{CP}^2 = \overline{CB}^2.$$

And, by the Corollary to the 26th Proposition,

$$CV \times CK = \overline{CB}^2.$$

Wherefore, $CV \times CK = \overline{CV}^2 + \overline{CP}^2,$

$$CK \times CV - \overline{CV}^2 = \overline{CP}^2,$$

$$(CK - CV)\, CV = \overline{CP}^2.$$

That is, $VK \times CV = \overline{CP}^2.$

Therefore, $CV : CP :: CP : VK.$

Now, by the foregoing Proposition,

$$PH : OV :: CV : CP.$$

And as $CP : VK :: CV : CP$,
$PH : OV :: CP : VK$.
Then, $OV : PH :: VK : PC$;
Or, $VO : VK :: PH : PC$.

Therefore, by the 6th Proposition of the Sixth Book of ELEMENTS, the triangles CHP and KOV are equiangular, and have those angles equal which are opposite to the homologous sides.

Therefore the alternate angles PCH and VKO are equal.

Wherefore, by the 27th Proposition of the First Book of ELEMENTS, CH and KO are parallel.

COR. II.—If, from the point in which the circumference of the alternate circle is cut by the aforementioned ordinate, there be taken, on its diameter, another point, at a distance equal to the difference between that diameter and the greater diameter of the ellipse, and from the alternate end of that ordinate, there be drawn to that other point a line; it shall also be parallel to the diameter drawn from the alternate end of the other ordinate.

From the point V, take the distance VI, equal to VK, and join YI.

YI shall be parallel to HL.

For, because OY is an ordinate, OV is equal to YV.

And, because the angles OVK and IVY are vertical angles, they are equal.

Therefore the triangles KVO and IVY have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another.

Wherefore, by the 4th Proposition of the First Book of ELEMENTS, the two triangles are equal, and the other angles are equal, each to each.

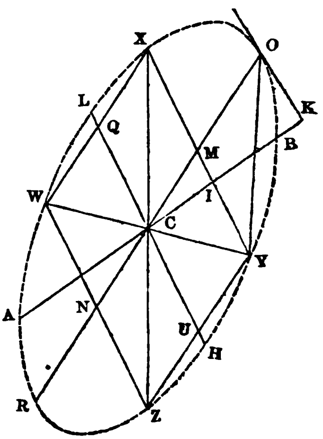
Therefore the angle VIY is equal to the angle VKO, and the angle VYI is equal to the angle VOK.

Wherefore, by the 27th Proposition of the First Book of Elements, IY and KO are parallel.

And KO has been shown to be parallel to CH.

Therefore, by the 30th Proposition of the First Book of Elements, YI is parallel to LH.

Cor. iii.—The straight line drawn from the end of the produced semi-diameter to the end of the ordinate drawn through the point in which that diameter is cut by the circumference of the alternate circle, is a tangent to the ellipse.



Produce KO indefinitely. KO is a tangent to the ellipse.

Produce YI to X, a point in the circumference of the ellipse, and, through the centre C, draw the straight lines XZ, YW, to the circumference of the ellipse.

Join WX, XY, YZ, and WZ.

Then, by the 4th Proposition of the Third Book of the Treatise, XZ and YW are diameters of the ellipse.

And, by the 2d Corollary of the 2d Proposition of the Fourth Book of the Treatise, WX, XY, YZ, and WZ, are four sides of a parallelogram.

And it is inscribed in the ellipse.

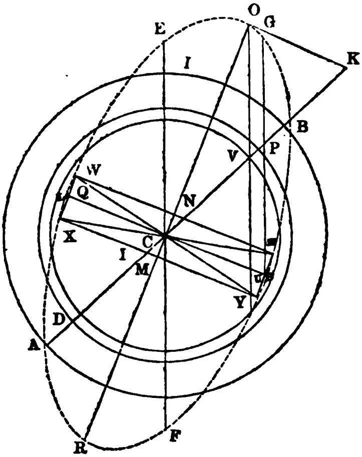
And, of these four sides of the parallelogram, XY is parallel to LH.

Wherefore, the opposite side, WZ, is parallel to LH.

Therefore, the other sides, WX and YZ, are cut by the diameter LH.

Let them be cut in the points Q and U.

Then, $QUYX$ is a parallelogram.



Wherefore, by the 34th Proposition of the First Book of Elements, QX and UY are equal to one another.

For a like reason, QW and UZ are equal to one another.

Then, because the angles QCX and UCZ are vertical angles, they are equal.

And, because CXQ and CZU are alternate angles, they are equal.

And the sides CX and CZ are equal, by the 5th Definition of the Third Book of the Treatise.

Therefore, by the 26th Proposition of the First Book of Elements, QX is equal to UZ.

Now, QX is equal to UY.

Therefore, UZ is equal to UY, and the line YZ is bisected in the point U.

In the same manner, the line WX may be shown to be bisected in the point Q.

Then, because the angles MCX and NCZ are vertical, they are equal.

And, because the angles MXC and NZC are alternate, they are equal, by the 29th Proposition of the First Book of Elements.

Wherefore, the angles CMX and CNZ are also equal.

And, because CX and CZ are equal, the other sides of the triangles, CMX, CNZ, are equal, and MX is equal to NZ.

In like manner, *MY* may be shown to be equal to *NW*.

And because the triangles *CNZ* and *CUZ* are upon the same base, *CZ*, and between the same parallels, *CU* and *NZ*, they are equal to one another, by the 37th Proposition of the First Book of Elements.

And, because the triangles *CNZ* and *CUZ* are equal, and that the angle *UCZ* is equal to the angle *CZN*; *CZ* is to *CU* as *CZ* is to *ZN*, by the 15th Proposition of the Sixth Book of Elements.

Therefore, *CU* is equal to *ZN*.

In the same manner, *CU* may be shown to be equal to *YM*.

But *MY* has been shown to be equal to *NW*.

Therefore *ZN* is equal to *NW*.

Wherefore *WZ* is bisected in the point *N*.

In the same manner *XY* may be shown to be bisected in the point *M*.

Therefore the two straight lines, *LH* and *OR*, in the ellipse, bisect the opposite sides of *WXYZ*, the parallelogram inscribed in it.

LH and *OR* are therefore conjugated diameters, by the 10th Proposition of the Fourth Book of the Treatise.

Now, the line *KO* is drawn through the end of one of these two conjugated diameters, and is parallel to the other.

Therefore, by the 3d Definition of the Fourth Book of the Treatise, *KO* is a tangent to the ellipse.

Cor. iv.—If half of the diameter common to the ellipse and the external circle be a mean proportional between the radius of one of the alternate circles and the semi-diameter produced, and an ordinate be drawn through the point in which the semi-diameter is cut by the circumference of that alternate circle; a straight line, drawn from the end of the produced semi-diameter through the end of the ordinate, shall be a tangent to the ellipse.

Thus, let CB be a mean proportional between CV and CK; the former, namely, CV, being the radius of one alternate circle STV, and the latter, CK, being CB produced; and let OY be an ordinate drawn through V, the point in which CB is cut by the circumference of STV, and let KO be joined and produced indefinitely; then KO is a tangent to the ellipse.

The demonstration of this Corollary is to be found in those of the preceding Corollaries, and especially in that of the third.

Schol.—When the tangent falls upon the end of the ordinate drawn through the point in which one alternate circle cuts the common diameter; the diameter parallel to that tangent falls upon the opposite end of an ordinate, drawn through the point in which the other alternate circle cuts the common diameter.

Thus, let the tangent KO fall upon O, the end of the ordinate OY, drawn through V, the point in which the alternate circle STV cuts the common diameter AB, and let LH be the diameter parallel to KO; then LH shall fall upon the opposite end of GH, the ordinate drawn through P, the point in which the other alternate circle, PDJ, cuts AB, the common diameter.

Cor. v.—If, from the alternate ends of two ordinates, drawn through the points in which the diameter, common to an ellipse and an external circle, is cut by two alternate circles, there be drawn two diameters to the ellipse; those diameters shall be conjugated.

Schol.—The truth announced in the above-stated 3d Corollary is the same as that set forth in the 3d Corollary of the 6th Proposition of the Second Book of the Conic Sections of the Marquis de l'Hospital.

ALTERNATE CIRCLES.

Section the Second.

According to the method hitherto used by mathematicians, the equations to the ellipse are quadratick. The reason of this is, that algebraists have been used to express the properties of the ellipse by signs and letters denoting the proportions that certain squares and parallelograms bear to each other.

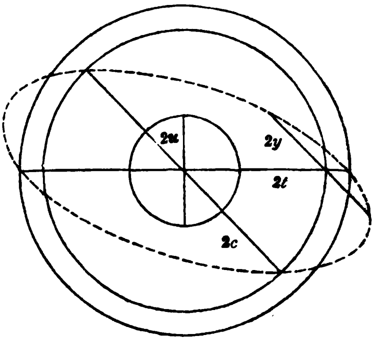
But, in the method now to be proposed, and resting upon the theory exhibited in the Treatise, the properties of the ellipse are to be expressed by signs and letters denoting the proportions that certain straight lines bear to each other. These will, at first, produce only simple equations.

Thus, to express in algebra the conclusion to which we come in the 15th Proposition of the Third Book of the Treatise, namely, that the diameter of the alternate circle is, to the ordinate, as the diameter of the ellipse to its conjugate, and calling the diameter of the alternate circle $2u$, the ordinate $2y$, the diameter of the ellipse $2t$, and its conjugate $2c$, we have

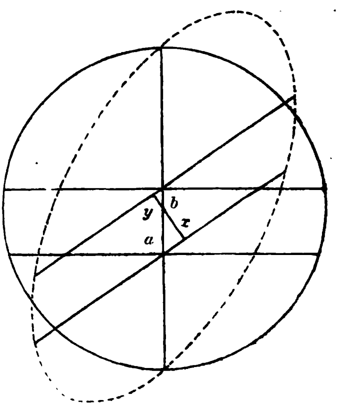
$$2u : 2y :: 2t : 2c,$$

or, $ty = cu$,

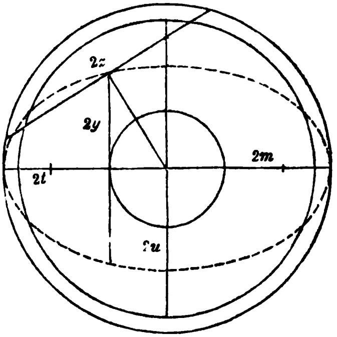
which is a simple equation.



Geometrical proportions between other straight lines, or between their sums, or their differences, are the results of the mode of reasoning employed in the Treatise on the ELLIPSE. These results may be expressed in algebra, and the consequent equations will be simple. Thus, because, by the 9th Proposition of the Third Book, if a circle and an ellipse have a common diameter, the distance of the centre from an ordinate of the ellipse is proportionate to its distance from the corresponding ordinate of the circle; we may consider the distances of the ordinates of the circle from its centre as given quantities, certain lengths taken on the diameter, and call them a and b; and we may consider the distances of the ordinates of the ellipse from the same centre as unknown, and call them x and y, respectively corresponding with a and b. Then, $x : a :: y : b$; that is, $ay = bx$, which is a simple equation.



In the 25th Proposition, the diameter of the alternate circle is to the chord of the external circle as the greater axis of the ellipse to the line which joins the focuses.



Calling the diameter of the alternate circle $2u$, the chord $2z$, the greater axis $2t$, and the line which joins the focuses $2m$, we find

$2u : 2z :: 2t : 2m$;

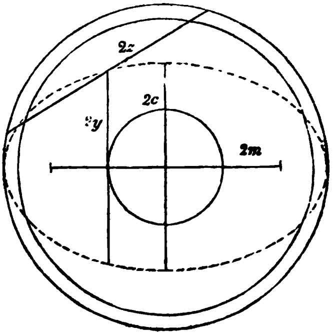
that is, $mu = tz$.

In the 1st Corollary to the same Proposition, the ordinate is to the chord as the lesser axis to the line which joins the focuses.

Calling the ordinate $2y$, the chord $2z$, the lesser axis $2c$, and the line which joins the focuses $2m$; we find

$2y : 2z :: 2c : 2m$;

or, $my = cz$.



Equations of higher dimensions may, however, be easily obtained, by treating such as the foregoing in a certain way. Or they will occur in analytical reasoning about the ellipse.

That such equations arise from treating alternate circles analytically, is evident from the solution of the Problems in the preceding Book.

And, with regard to the ellipse, we may, even by proceeding a little further than we have here done, convert some of these simple equations into precisely the same quadraticks as are found in the usual way.

For example, having, from the 15th Proposition of the Third Book of the Treatise, deduced the equation $cu = ty$, and the proportion, $t : c :: u : y$, we find, by the 22d Proposition of the Sixth Book of ELEMENTS,

$$tt : cc :: uu : yy.$$

Now, calling the absciss x, we find that x represents the radius of the alternate circle in which the absciss is.

Then, by the definition of alternate circles,

$$4xx + 4uu = 4tt,$$

that is, $$uu = tt - xx,$$

then, $$tt : cc :: tt - xx : yy.$$

Wherefore, $yy = cc - \frac{ccxx}{tt}$, which is the same formule as one of those in the 41st paragraph of the Treatise on the Conic Sections by the Marquis de l'Hospital.

The use of converting simple equations into some of higher dimensions is, to find out the constitution of the latter, and to see how they may be reduced.

So, by multiplication, we arrive at the knowledge of division. And so, by involution, we may find the way to evolution.

Thus, in the present case, going back to where we began, that is, taking the opposite course, we find that the equation $yy = cc - \frac{ccxx}{tt}$ comes to the geometrical proportion, $tt : cc :: tt - xx : yy$;

and as $tt - xx$ represents the difference between the squares of two lines, that are, in fact, the radiuses of two concentric circles, we may, by describing a circle alternate to the least of these, obtain one, the square of whose radius is equal to the difference between the squares of the two lines in question. We may then substitute it for that difference, and thus obtain the proportion,

$$tt : cc :: uu : yy,$$

which comes to, $t : c :: u : y;$

or, $cu = ty,$

a simple equation.

Now, as in every equation that expresses the properties of the ellipse, in the common method, there are two terms with contrary signs, thus expressing the difference between the squares of two lines, of which one is a fixed, and the other a variable quantity; and as by the present process there may be substituted for that difference, and equal to it, the square of one variable line, the number of the terms in the equation may thus be reduced. And then, as in some cases, the equation may be turned into a proportion consisting of four rational terms, the roots of those terms may be extracted, and the equation may be simplified.

Thus, taking the figure belonging to the 25th Proposition of the Third Book of the Treatise, and calling TV $2u$, IK $2z$, AB $2t$, and MN $2m$, we have, by that Proposition, the proportion,

$$2u : 2z :: 2t : 2m,$$

$$uu : zz :: tt : mm.$$

Now, CP is the absciss, and may be called x. It is also the radius of the internal circle. The diameter of that circle will therefore be $2x$.

Then the square of the diameter of the alternate circle is equal to the difference between the square of the greater axis and the square of the diameter of the internal circle; that is, $4tt - 4xx = 4uu$. Wherefore,

$$4tt - 4xx : 4zz :: 4tt : 4mm;$$

or, $mmtt - mmxx = ttzz,$

a quadratick equation.

How this may be reduced to the utmost simplicity has been already shown in the former case. But, in both cases, the Proposition had been already demonstrated; and we have therefore begun with the result. This has facilitated the operations. If we had begun the demonstration algebraically, there might have occurred, in the course of it, a case of much more complexity, one in which the preceding mode of reduction might have been much more wanting.

This complexity, however, may not take place if the demonstration be begun on the principle of alternate circles. That principle tends to introduce simplicity. Thus, taking the 15th Proposition of the Third Book of the Treatise, and calling AB $2t$, EF $2c$, TW $2u$, and GH $2y$, we have,

$$DR = 2t,$$

and

$$2t : ON :: 2c : 2y,$$

by the 10th Proposition.

And again,

$$ON = 2u;$$

therefore,

$$2t : 2u :: 2c : 2y.$$

Then,

$$cu = ty,$$

and

$$u : y :: t : c.$$

That magnitudes have the same ratio to one another which their equimultiples have, and that if four straight lines be proportionals, their squares shall also be proportionals; and if four squares be proportionals their sides shall be proportionals; should always be recollected in what is to follow.

The effect by which quadratick equations, which consist of three terms at least, are now reduced to two, and are thus rendered reducible to simple equations, is brought about not merely by substituting one expression for another. A different property of the ellipse is simultaneously introduced. Thus, if the diameter of the ellipse be called $2t$, and its conjugate $2c$, the absciss x, and the semi-ordinate y; then, because the square of the diameter is to the square of its conjugate, as the difference between the square of half the diameter and that of the absciss is to the square of the semi-ordinate, the proportion is expressed thus,

$$4tt : 4cc :: tt - xx : yy.$$

But, in the method now proposed, the diameter of the ellipse is the same with the diameter of the external circle, the absciss is the same with the diameter of one alternate circle, and, therefore, the difference between the squares of their radiuses is equal to the difference between the square of half the diameter of the ellipse and the square of the absciss. Now the difference between the squares of the radiuses above-mentioned is also equal to the square of the radius of the alternate circle, and may therefore, in algebra, be substituted for it. But, though equal in quantity, it is a different thing. It is the same in quantity, but not in figure. It is the same in algebra, but not in geometry. It therefore expresses a different property of the ellipse.

The reduction of an equation is very different from its solution. The reduction of an equation is the diminution of its dimensions. To solve it is to show the value of its unknown quantity in quantities that are given.

The processes of reduction and solution differ as widely, or still more widely, than the objects.

Reduction (as has been shown) may be effected by substituting one term for two.

Solution may be effected by substituting two terms for one.

Thus, if the equation be

$$xx - (a + b)x + ab = 0;$$

then, if we make $x = y + \frac{a + b}{2}$, and substitute for

x its value thus assumed, the equation becomes

$$yy - \frac{(a+b)^2}{4} + ab = 0,$$

in which the second variable term no longer appears, as it has vanished in the process.

By proceeding towards the solution of this equation, we find,

$$yy = \frac{aa - 2ab + bb}{4}, \text{ whence}$$

$$y = \pm \frac{a-b}{2} = + \frac{a-b}{2}, \text{ or } - \frac{a-b}{2}.$$

Therefore,

$$x = y + \frac{a+b}{2} = + \frac{a-b}{2} + \frac{a+b}{2} = \frac{2a}{2} = a,$$

or,

$$x = y + \frac{a+b}{2} = - \frac{a-b}{2} + \frac{a+b}{2} = \frac{2b}{2} = b.$$

The foregoing theory gives to the ellipse a new characteristick, inasmuch as it may now be expressed by a simple equation, and does not require the second powers of the variable quantities.

The variable quantities expressed by the simple equation are, indeed, not both the same as those expressed by the quadratick equation in the former method, in which one variable quantity was the absciss and the other the ordinate. In the present method, there is an absciss and an ordinate, but they are not both expressed. The ordinate is expressed by the equation. The absciss is the semi-diameter of one alternate circle. But the other quantity, expressed as corresponding to the ordinate, is the diameter of the other alternate circle.

The relation that exists between the formule hitherto used for expressing the nature of the ellipse with reference to its diameters, and that now proposed, may be shown in the following manner:

The expression $tt - xx : yy :: 4tt : 4cc$ means, that the difference between the square of one semi-diameter of the ellipse and the square of an absciss, is to the square of half the corresponding ordinate as the square of the whole diameter is to the square of its conjugate.

But, according to the 15th Proposition of the Fifth Book of ELEMENTS, the foregoing expression may be converted into

$$4tt - 4xx : 4yy :: 4tt : 4cc;$$

which means, that the difference between the square of the diameter and the square of twice the absciss is to the square of the corresponding ordinate as the square of the same diameter is to the square of its conjugate.

Now, the diameter of the ellipse is equal to the diameter of the external circle.

Twice the absciss is equal to the diameter of one of the alternate circles.

Therefore, the difference between the squares of these two lines is equal to the square of the diameter of the other alternate circle.

Substituting this quantity for the difference to which it is equal, we come to the following Proposition:—

The square of the diameter of the alternate circle is to the square of the ordinate as the square of the diameter of the ellipse is to the square of its conjugate.

Wherefore, by the 22d Proposition of the Sixth Book of ELEMENTS, we have the following:—

The diameter of the alternate circle is to the ordinate as the diameter of the ellipse to its conjugate.

Which exactly agrees with the 15th Proposition in the Third Book of the Treatise; and, calling the diameter of the alternate circle $2u$, and the ordinate $2y$, we come to

$$2u : 2y :: 2t : 2c;$$

or,

$$u : y :: t : c.$$

Still this simplicity is, in reality, not so great as at first it appears to be. Modern analysis expresses the nature of curves principally by the relation that subsists between the absciss and the ordinate, which always meet at some point. In the method now proposed, the absciss disappears. Some operations may still be performed upon the curve by the use of the simple equation. But, to make these intelligible to the modern analyst, the absciss must be restored to its place. In doing this, we find that the diameter of the alternate circle is replaced by a surd; that is, by the square root of the difference between two squares. One of these squares is that of the diameter of the ellipse. The other is that of twice the absciss. Algebraically we may say,

$$2u = \sqrt{4tt - 4xx} = 2\sqrt{tt - xx}.$$

When we substitute for u its value, we introduce the absciss; but it returns with a complexness that places the ellipse amongst lines of the second order, and at the same time enables us to lay it down on the same principle as other curves that are called geometrical.

The same thing occurs if we advert to the principle contained in the 25th Proposition of the Treatise on the ELLIPSE.

There, if we call the diameter of the alternate circle $2u$, the chord $2z$, the greater axis $2t$, and the line which joins the focuses $2m$, we come to the expression,

$$2u : 2z :: 2t : 2m;$$

wherefore, $$z = \frac{mu}{t},$$

than which no equation can be, apparently, more simple.

But, substituting for $2u$ its value $\sqrt{4tt - 4xx}$, we have,

$$\sqrt{4tt - 4xx} : z :: 2t : 2m;$$

whence, $$z = \frac{m\sqrt{tt - xx}}{t}.$$

Then, calling the lesser axis $2c$, and the ordinate $2y$, by the 1st Corollary of the same Proposition we have,

$$2y : \frac{2m\sqrt{tt - xx}}{t} :: 2c : 2m;$$

or, $$4mtty = 4cmt\sqrt{tt - xx};$$

which comes to $$yy = cc - \frac{ccxx}{tt};$$

which is the very formule in the 41st paragraph of the Treatise that the Marquis de l'Hospital has written on the Conic Sections.

In the equation, $tz = mu$, the absciss and the ordinate both disappear. One of the variable quantities is the diameter of an alternate circle; the other is a chord of the external circle. But these quantities vary in magnitude conformably to the variations of the absciss and the ordinate.

The common equations to the ellipse express a relation between two straight lines that are both variable, and of which each is terminated, at one end, by the point at which they meet. This relation is denoted by introducing the squares of two other lines which are fixed and invariable. These squares constantly bear the same proportion to each other, although the variable lines do not. But their squares bear to each other the proportion which is borne to each other by the square of one of the variable lines and the difference between the square of one of the fixed lines and the square of the other variable line. This proportion is expressed by the common equation.

The method of alternate circles presents a line of which the square is precisely equal to the foregoing difference. The square of that line may therefore be substituted for that difference in the common equation. This substitution introduces into the equation itself a great deal of simplicity; for thus the equation is made to consist of four complete squares, of which the roots may be extracted. The roots bear to one another a geometrical proportion, and form a simple equation, or one of the first degree. But this substitution deprives the equation of one important characteristick; that of expressing the relation between the two variable lines. For, upon the substitution taking place, one of those lines disappears. And the line that is introduced is of quite another character, appears in another place, and bears a different relation to the other quantities in the proportion, or the equation that is derived from it.

However, the result of the simple equation may be used with much more facility than can be that of a quadratick; and, if the equation of greater dimensions should again be wanted, it may be restored by reversing the process by which it had been reduced.

Another kind of simplicity may be obtained by the same expedient. Thus, if the equations

$$tt - 2tz + zz = yy + mm - 2mx + xx,$$

and $tt + 2tz + zz = yy + mm + 2mx + xx,$

be to be added together, their sum is

$$2tt + 2zz = 2yy + 2mm + 2xx,$$

which comes to

$$tt - xx = yy + mm - zz.$$

Now, the substitution of uu for $tt - xx$ gives

$$uu = yy + mm - zz,$$

$$uu - yy = mm - zz,$$

$$(u + y)(u - y) = (m + z)(m - z)$$

$$u + y : m + z :: m - z : u - y.$$

The foregoing equations are taken from the 1st Proposition of the Second Book of the Conic Sections of the Marquis de l'Hospital, in which he shows that $z = \frac{mx}{t}$.

And in the 2d Proposition he shows that, by putting, in the place of z, its value in the equation,

$$tt \pm 2tz + zz = yy + mm \pm 2mx + xx,$$

it becomes

$$ttyy = t^4 - ttxx - mmtt + mmxx,$$

or, $$ttyy + mmtt - mmxx = t^4 - ttxx,$$

or, $$ttyy + (tt - xx)\ mm = (tt - xx)\ tt.$$

Now, substituting for $tt - xx$ its value uu, we have

$$ttyy - ttuu = -mmuu,$$

$$-tt\ (yy - uu) = mmuu,$$

$$yy - uu : uu :: mm : -tt.$$

The Marquis de l'Hospital makes the centre of the ellipse the origin of the absciss. He calls half the greater axis t, half the line that joins the focuses m, the absciss x, the semi-ordinate y, and a portion of the axis, commencing at its extremity, and equal to a line drawn from the neighbouring focus to the end of the semi-ordinate, z.

From hence arises the equation,

$$tt - 2tz + zz = yy + mm - 2mx + xx.$$

By substituting for z its value, and for the difference between the square of half the greater axis and the square of the absciss, reckoned from the centre, the square of the alternate circle, we arrive at the equation,

$$tt\ (uu - yy) = mmuu.$$

Now, if we substitute for u, $\sqrt{ax - xx}$; for m, $\frac{a - c}{2}$; and for t, $\frac{a}{2}$; the proportion,

$$uu - yy : uu :: mm : tt, \text{ becomes}$$

$$ax - xx - yy : ax - xx :: aa - 4ac + 4cc : aa.$$

Then, by the 17th Proposition of the Fifth Book of Elements,

$$-yy : ax - xx :: -4ac + 4cc : aa,$$

$$-aayy = (-4ac + 4cc)\ (ax - xx),$$

$$aayy = (4ac - 4cc)(ax - xx),$$

$$yy = \frac{4ac - 4cc}{aa} \times (ax - xx);$$

which is the formule of M. Bézout, in his course of Mathematics.

But M. Bézout makes the extremity of the axis the origin of the absciss. He calls the axis a, the absciss x, the ordinate y, and the distance between the extremity of the axis and the nearest focus c; and from these, by a process of reasoning, he comes to the equation now stated.

By the use of the alternate circles, we have here a kind of simplicity, although it does not lead us to a simple equation. It leads to a proportion in which three of the magnitudes are complete squares, the other consisting of the difference between two.

The simplicity of an equation depends on the nature and number of the quantities expressed. Many dimensions require high powers; and many numbers require many terms. The lower the powers, and the fewer the terms, the greater the simplicity of the equation.

When one term may be fairly substituted for the sum of two, or for the difference between two, a degree of simplicity may be introduced.

With regard to Propositions, the case is very much otherwise. With them, number conduces to simplicity. A final result may sometimes be obtained by means of several propositions and demonstrations, with much more simplicity than by a single one.

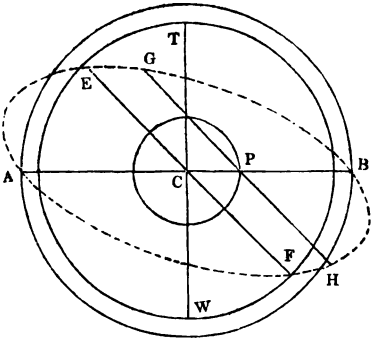
Thus, the 6th Proposition of the Second Book of the Conic Sections of the Marquis de l'Hospital is, that the square of a semi-ordinate to a diameter of an ellipse is to the rectangle contained by the parts of that diameter, as the square of its conjugate is to the square of the whole diameter.

He, indeed, says " ordinate," but means what we call " semi-ordinate."

The 15th Proposition of the Third Book of the Treatise, is, that if a diameter common to an ellipse and an external circle be divided by an ordinate, the diameter of the alternate circle shall be to the ordinate, as the diameter of the ellipse to its conjugate.

Now if, for the square of the semi-diameter of the alternate circle, we substitute its equal; that is, the rectangle contained by the parts of the diameter, the 6th Proposition of the one Book is immediately deducible from the 15th of the other.

In the 15th Proposition, the diameter of the alternate circle is $2\,CT$, the ordinate is $2\,PG$, the diameter of the ellipse is AB, and its conjugate is EF.



And the result is,

$$TW : GH :: AB : EF.$$

$$\text{Or,}\quad 2\,CT : 2PG :: AB : EF.$$

$$\text{Wherefore,}\quad CT : PG :: AB : EF.$$

$$\text{And}\quad \overline{CT}^2 : \overline{PG}^2 :: \overline{AB}^2 : \overline{EF}^2.$$

$$\text{But,}\quad \overline{CT}^2 = AP \times PB.$$

Wherefore,

$$AP \times PB : \overline{PG}^2 :: \overline{AB}^2 : \overline{EF}^2.$$

$$\text{Or,}\quad \overline{PG}^2 : AP \times PB :: \overline{EF}^2 : \overline{AB}^2.$$

That is, the square of the semi-ordinate is to the rectangle contained by the parts of the diameter, as the square of the conjugate to the square of the whole diameter.

But then the result has been arrived at by means of many previous propositions, and simplicity may be considered as obtained by means of prolixity.

The same principle may be carried further; and, in laying down the curve of the ellipse, we may make use of two equations.

An absciss and an ordinate are wanted for laying down a curve.

Now, when the ordinate passes through the point in which one alternate circle cuts the diameter of the ellipse, the absciss is the radius of that circle; and that radius is the square root of the difference between the squares of the radiuses of the two other circles.

Thus, if we call the diameter of the alternate circle $2u$, its radius will be u. If the diameter of the ellipse be $2t$, the radius of the external circle will be t.

Then, if the absciss be x, we have the equation,

$$xx = tt - uu, \text{ or, } x = \sqrt{tt - uu}.$$

Now, because $ty = cu$, $y = \frac{cu}{t}$.

The usual method gives one equation and two variable quantities.

The method now proposed gives two equations and three variable quantities.

But u may be anything, provided it be less than t. And the value of u determines the values of x and y, which may be deduced from it with much ease. Their values determine the circumference of the curve.

And, in order to describe it, there is no necessity for describing the circles. The knowing the values of their radiuses is sufficient.

But, in laying down curves, the fixing the absciss arbitrarily may be most convenient.

This may be easily done without departing from the proposed method. For, as $xx = tt - uu$, $uu = tt - xx$, and $u = \sqrt{tt - xx}$. Now, x is supposed to be determined at pleasure; wherefore u is immediately known. Then the ordinate is also known, by means of the simple equation, $y = \frac{cu}{t}$.

THE END.

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